mathematics for neuroscience

pablo catalán

uc3m | Universidad Carlos III de Madrid
Departamento de Matemáticas

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Colophon
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Cover image
"Splendid Working (Functioning Arises from a Pure Heart)", calligraphy by Keidō Fukushima (1933-2011).

Contents

Introduction

This course is an introduction to Mathematics for students in the Neuroscience Degree. The range and depth of math that could be taught in this course is very wide, so I will try to focus on those concepts that will be most useful for the degree. Therefore, the course will cover basic differential and integral calculus, linear algebra and differential equations, with a focus on these from the perspective of dynamical systems, which I believe will be most useful for students dealing with nonlinear models. Many more topics could be included, but time is (sadly) limited. If, after reading this book, you realize a newfound love for mathematics, please keep studying them.

For Chapters 1 to 5 I have copied a lot of material from the book "Differential and Integral Calculus of a Single Variable" by my colleague and mentor José A. Cuesta.

Chapter 6 is copied almost verbatim from Steven Strogatz's excellent "Nonlinear Dynamics and Chaos".

For the Linear Algebra chapters I have taken most of the material from Lay, "Linear Algebra and its Applications".

I took many examples and exercises from Claudia Neuhauser and Marcus Roper's "Calculus for Biology and Medicine" which, in fact, covers all the sections in this course, and many more.

ChatGPT helped me format the excerpted parts of th books I wanted to include into LaTeX.

I'm grateful to the students that pointed out inconsistencies and who pointed out areas where the topics could be better explained. I hope to keep improving the notes as time goes on.

Part I. Differential and Integral Calculus

Functions | 1

1.1 An Introduction to Functions

A mathematical **function** is a rule that assigns an element from a given set to an element of another set. In other words, a function is a mathematical object that returns an output when you hand it an input. The set of possible inputs of a given function is called the **domain** and the set of possible outputs is called the **range** or **image**[1] . The usual notation for functions is $y = f(x)$, where f represents the rule that assigns the output y to the input x.

1: Sometimes this is also called the codomain, in case you see it in another books.

Let's see how this works with a well-known function: $f(x) = x^2$. What is the domain of this function? If we don't specify it, it could be many things: the set of all matrices with positive numbers, or the set of all complex numbers, or the set of all polynomials of fifth degree[2] .

2: If you don't understand some of these words, don't worry, we'll see them as we advance in the course.

In this course, we will work mostly with functions of **real numbers**, which we denote with the letter $\mathbb{R}$. The real numbers are very interesting and if we had time we could talk about how very weird they are. But, for our purposes, let's just say that real numbers are those we can write in decimal form, like $3.141592653\ldots$, $2.718281828\ldots$, $1.618033988\ldots$ or $0.999999\ldots$. Those real numbers that have a finite or periodic decimal expression are called **rational numbers**, $\mathbb{Q}$, because we can always express them as a fraction. The rest are called **irrational numbers** and their decimal expressions are only approximations to their true value.

We represent the real numbers on a line, going from $-\infty$ to ∞, and usually marking where 0 is, the separation between positive and negative numbers.

When defined on the real numbers, the function $f(x) = x^2$ becomes a real function from some domain $D \subset R$ to the reals:

$$\begin{aligned} f : D &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = x^2 \end{aligned} \tag{1.1}$$

What is the set D? In other words, what are the valid inputs for x^2? All the real numbers, since the expression makes sense for them. Given any number, positive or negative, x^2 returns its square. Now, this output is always non-negative, and so the range of $f(x) = x^2$ is the non-negative real line.

Example 1.1.1 Other examples of functions:

1. $y = |x|$ represents the rule $f(x) = |x|$ that maps each number x to its absolute value.
2. The function
$$f(x) = \begin{cases} x^2 & x \leq 2, \\ x^3 - 3 & x > 2, \end{cases}$$

maps all real numbers smaller than or equal to 2 to their square, and those larger than 2 to their cube minus 3.

The usual way to represent functions is to plot their **graph**. For a function $f : A \subset A \to \mathbb{R}$, we need to draw this in a plane, where the horizontal axis (the x axis) represents the input and the vertical axis (the y axis) represents the output. For a given function, we need to plot every point $(x, f(x))$ for all $x \in A$.

If a function does not repeat outputs for two different inputs, we say it is **injective** or **one-to-one**. For instance, $f(x) = x + 5$ is injective. But $f(x) = x^2$ isn't. If a function is injective, the equation $y = f(x)$ has either no solution or a unique solution.

On another hand, if a function covers all of the range (in these examples, that means $\mathbb{R}$) we say it is **surjective** or **onto**. Again, $f(x) = x + 5$ is surjective, but $f(x) = x^2$ isn't.

If a function is both injective and surjective it is called **bijective**. A bijective function is a perfect correspondence between to sets.

A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

A function is **bounded** if there exists $M > 0$ such that $|f(x)| \leq M$ for all x in its domain.

A function is **monotonically increasing** if for every x, y in its domain such that $x < y$ it satisfies $f(x) \leq f(y)$, and is **monotonically decreasing** if $f(x) \geq f(y)$. We say it is **monotonic strictly increasing/decreasing** if inequalities are strict. (Note that a constant is both monotonically increasing and decreasing, but not strictly.)

1.2 Operations with functions

Algebraic operations

Let $A, B \subset \mathbb{R}$ and consider the two real functions

$$\begin{aligned} f : A &\longrightarrow \mathbb{R} & \qquad g : B &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) & x &\longrightarrow y = g(x) \end{aligned} \tag{1.2}$$

With these two functions we can perform the following algebraic operations:

(i) **Addition:** If $C = A \cap B$ —where both functions are defined—,

$$\begin{aligned} f + g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) + g(x) \end{aligned} \tag{1.3}$$

(ii) **Product:** If $C = A \cap B$,

$$\begin{aligned} fg : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)g(x) \end{aligned} \tag{1.4}$$

(iii) **Quotient:** If $C = A \cap B'$, where $B' \equiv \{x \in B \,:\, g(x) \neq 0\}$,

$$\begin{aligned} f/g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x)/g(x) \end{aligned} \tag{1.5}$$

For instance, if $f(x) = x + 5$ and $g(x) = x^2$, the sum is a new function $h(x) = f(x) + g(x)$, in this case $h(x) = x^2 + x + 5$. Similarly, we can do this with products of functions and quotients.

Proposed Exercise 1.2.1 Give one example for product and quotient of functions, and identify the corresponding domains and ranges.

Compositions

A more involved operation is the **composition** of two functions. It is defined as

$$\begin{aligned} f \circ g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f\big(g(x)\big) \end{aligned} \tag{1.6}$$

In this case, the domain is not so simple to obtain. For $f \circ g$ to be defined x must belong to B, for $g(x)$ to be well defined, so $C \subset B$. But in order to evaluate $f\big(g(x)\big)$, the number $g(x) \in A$. Therefore

$$C = \{x \in B \,:\, g(x) \in A\}. \tag{1.7}$$

Even if A and B are simple sets, C may be much more involved:.

Composition is a noncommutative operation, i.e., $f \circ g \neq g \circ f$.

It is, however, associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$. We can thus define multiple compositions, like $f \circ g \circ h \circ w = f(g(h(w(x))))$, without ambiguity.

Inverses

We can introduce the identity function $\mathrm{Id}(x) = x$. Given a function $f : A \longrightarrow \mathbb{R}$, its **inverse** would be a function $f^{-1} : f(A) \longrightarrow \mathbb{R}$ such that $f \circ f^{-1} = f^{-1} \circ f = \mathrm{Id}$. The idea is that if f maps x to y, its inverse f^{-1} maps y back to x.

Not all functions have an inverse that is defined all over their image $f(A)$. For an inverse to exist the equation $x = f(y)$, for a given $x \in f(A)$, must have a unique solution: in other words, f must be injective.

For those functions that are not injective in their domain A, we might be able to define several inverses by constraining the domain to any subset where they are made injective. Thus, noninjective functions may have several inverses.

Example 1.2.1 Let $f(x) = x^2$. Its domain is $\mathbb{R}$, but this function is not injective in its domain. However, we can constrain the domain to be $[0, \infty)$. In that case $f(x)$ is injective and we can obtain the inverse function by finding the unique solution of the equation $x = f(y) = y^2$, where $0 \leq y$. Clearly this solution is $y = \sqrt{x}$, therefore, within $[0, \infty)$, the inverse of f is $f^{-1}(x) = \sqrt{x}$.

Note that we might alternatively chosen the domain to be $(-\infty, 0]$, where the function f is again injective. However now the solution of $x = y^2$ with $y \leq 0$ is $y = -\sqrt{x}$. So another inverse of f is $f^{-1}(x) = -\sqrt{x}$.

The graph of $f^{-1}(x)$ can be obtained from that of $f(x)$ as the mirror image with respect to the line $y = x$.

Remark 1.2.1 BEWARE!! Never confuse $f^{-1}(x)$ with $f(x)^{-1} = 1/f(x)$, the reciprocal of f. In the case $f(x) = x + 5$, its inverse $f^{-1}(x) = x - 5$, whereas $(f)^{-1} = 1/(x + 5)$.

1.3 Elementary Functions

Let's introduce the most common functions used in nearly all mathematical problems, which we call **elementary functions**.

Polynomials

These are functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{1.8}$$

where $a_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, n$. The largest power, n, is called the *degree* of the polynomial. Constants are polynomials of degree 0. Given the operations that define them, the domain of any polynomial is $\mathbb{R}$.

Rational functions

They are defined as quotients of two polynomials, namely

$$f(x) = \frac{P_n(x)}{Q_m(x)}. \tag{1.9}$$

The domain of both polynomials is $\mathbb{R}$, but $Q_m(x)$ may be zero at some points, where the quotient will thus not be defined. Hence the domain of $f(x)$ is $\{x \in \mathbb{R} : Q_m(x) \neq 0\}$.

Trigonometric functions

The two basic trigonometric functions are the sine ($\sin x$) and the cosine ($\cos x$). In terms of them we can define also the tangent and cotangent:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{1.10}$$

The geometric definition of these functions, based on the unit circle, is described in Figure 1.1.

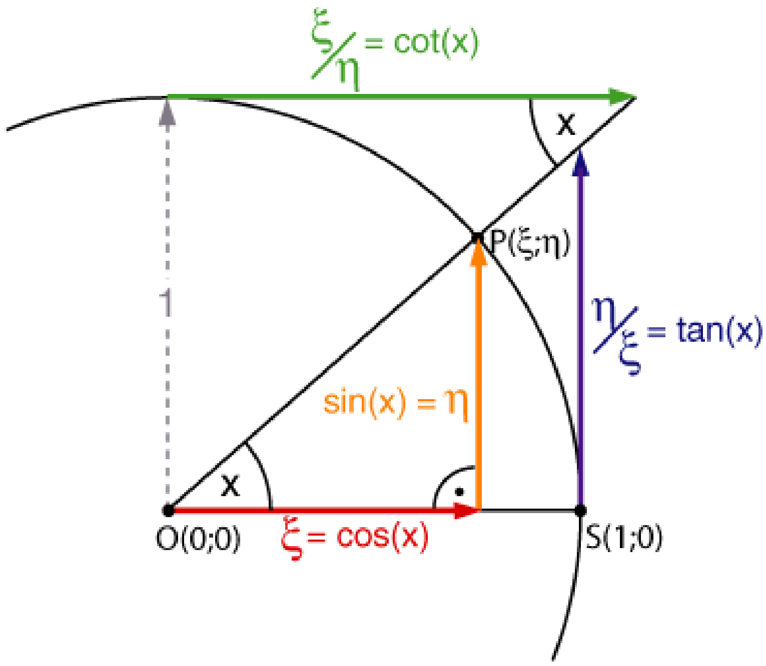


Figure 1.1: Geometric definition of $\sin x$, $\cos x$, $\tan x$, and $\cot x$.

There are two more trigonometric functions, although less common than the previous ones, namely the secant ($\sec x$) and the cosecant ($\csc x$):

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}. \tag{1.11}$$

The graphs of $\sin x$ and $\cos x$ are plotted in Figure 1.2. Those of $\tan x$ and $\cot x$ in Figure 1.3.

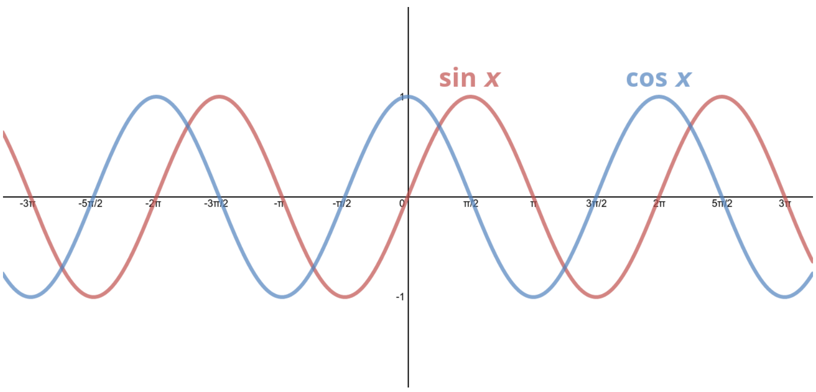


Figure 1.2: Plot of $\sin x$ and $\cos x$.

Given their geometric definitions, all these functions are related by geometric identities. The main ones are listed in Table 1.1.

Example 1.3.1 Periodic functions, like the trigonometric ones, are functions that "repeat" after some values. In mathematical terms, $f(x + c) = f(x)$, and the smallest c for which this is true is called the **period** (what is the period of the sine and cosine?). They are clearly not injective. Take $\sin x$, for instance. However, an interval where it is injective is $[-\pi/2, \pi/2]$, and so we can obtain the inverse of this

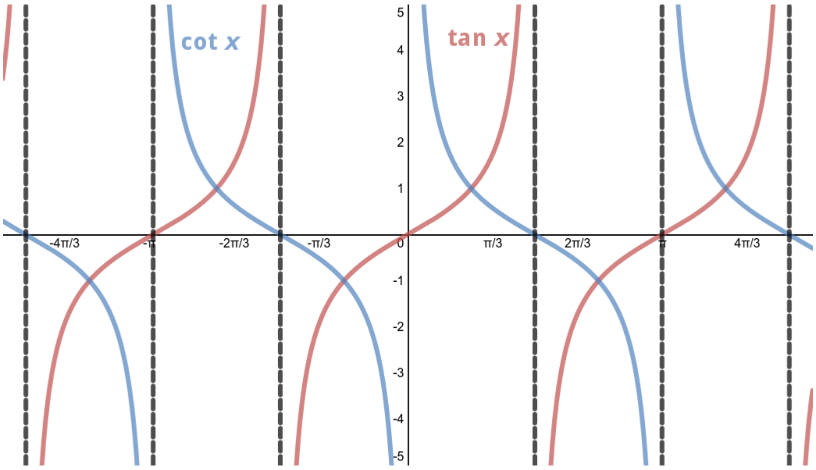


Figure 1.3: Plot of $\tan x$ and $\cot x$.

Table 1.1: Some important trigonometric identities.

Trigonometric identities
$\cos^2 x + \sin^2 x = 1$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
$1 + \tan^2 x = \sec^2 x$

function within this interval: the *arc sine:* $\sin^{-1} x = \arcsin x$. But we might have taken the interval $[\pi/2, 3\pi/2]$, for instance. In that case the inverse would be different: $\sin^{-1} x = \pi - \arcsin x$. Or in the interval $[3\pi/2, 5\pi/2]$ the inverse would be $\sin^{-1} x = 2\pi + \arcsin x$.

Similarly, $\arccos x = \cos^{-1} x$ when the domain of $\cos x$ is taken to be $[0, \pi]$, and $\arctan x = \tan^{-1} x$ when the domain of $\tan x$ is taken to be $(-\pi/2, \pi/2)$.

Exponential

This is the function defined as $f(x) = e^x$. The constant e appearing in this definition is the irrational number introduced by Euler

$$e = 2.71828182845904523536028747135266249775724709369995957\ldots$$

We will encounter the exponential in many of the problems we will explore later in the course, especially those dealing with differential equations.

The properties of the exponential are:

1. Its domain is $\mathbb{R}$.
2. $e^x > 0$ for all $x \in \mathbb{R}$.
3. It is monotonic strictly increasing —hence injective.
4. $e^0 = 1$.
5. $(e^x)^a = e^{ax}$ for any $a \in \mathbb{R}$.
6. $e^{x+y} = e^x e^y$.

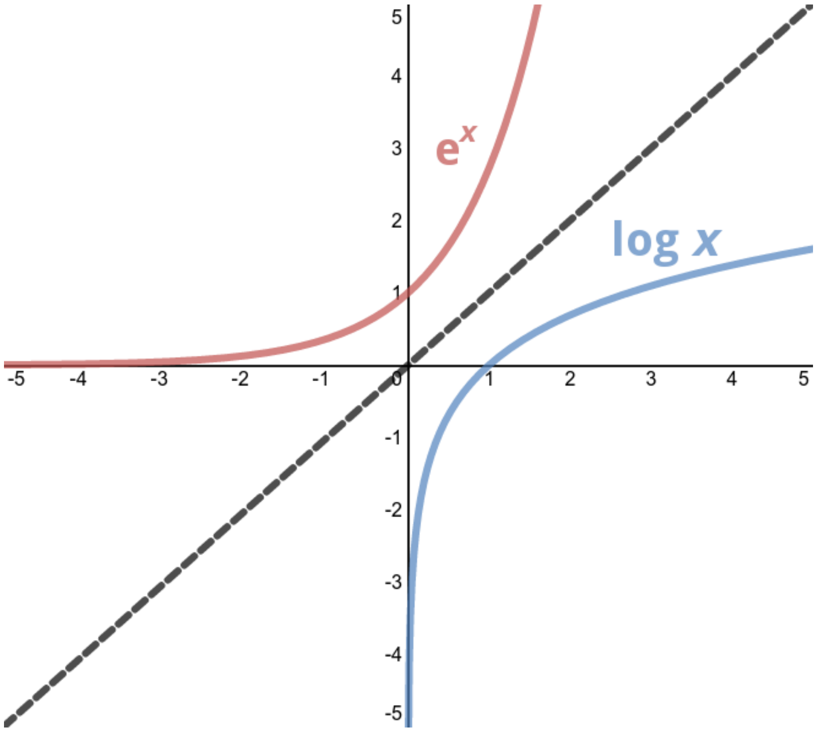


Figure 1.4: Plot of e^x and $\log x$.

7. $e^{-x} = 1/e^x$.

A plot of the exponential function is shown in Figure 1.4.

Logarithm

This is the inverse of the exponential. If $y = \log x$ it means that $x = e^y$. Its plot can be seen in Figure 1.4 to mirror that of the exponential with respect to the line $y = x$.

Remark 1.3.1 Along these notes, whenever we write $x = \log y$ we mean that x is the solution of the equation $e^x = y$, in other words, log of a number is the exponent to which we need to rise e in order to obtain that number. In particular $\log 1 = 0$ and $\log e = 1$.

The main properties of the logarithm (derived from those of the exponential) are the following:

1. Its domain is $(0, \infty)$.
2. Its image is $\mathbb{R}$ —hence it is surjective.
3. It is monotonic strictly increasing —hence injective.
4. $\log 1 = 0$.
5. $\log(x^a) = a \log x$.
6. $\log(xy) = \log x + \log y$.
7. $\log(x/y) = \log x - \log y$.

1.4 Limit of a function

Functions are defined for every single point of their domains. However, differential calculus has to do with the behaviour of functions "around" points, not just at them. The limit of a function is a way to characterise that behavior. The idea is to know what value the function is approaching as we get closer and closer to a certain point a (not necessarily in the domain of the function).

Example 1.4.1 Consider the function $f(x) = x^2$ and the point $a = 2$ (in the domain). As we take values of x closer and closer to 2, the output gets closer and closer to 4. This can be shown formally, but for our purposes it will be enough to understand the qualitative idea. We write

$$\lim_{x \to 2} x^2 = 4.$$

Example 1.4.2 The previous example might suggest that calculating a limit could be as simple as evaluating $f(a)$. To show that this is not always the case consider the function

$$f(x) = \frac{x-1}{x^2-1},$$

a rational function whose domain is $\mathbb{R} - \{1\}$. What happens as we get closer to 1? Using the calculator, we can see that $f(0.9) = 0.526$, $f(0.99) = 0.5025$, $f(0.999) = 0.5002$ and $f(0.9999) = 0.50002$. Again, this can be proven rigorously, but you get the idea. We write

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

even though 1 is *not* in the domain of f (hence $f(1)$ does not even exists).

If, as $x \longrightarrow a$, a function grows without limit, we say that the limit of f at a is infinite.

Proposed Exercise 1.4.1 What is the limit of $f(x) = 1/(x-1)$ as x approaches 1?

1.5 Continuity

Those functions whose limit at a point a of their domain coincides with the value of that function at that point play a very special role in calculus. They mainly coincide with those functions whose graph "can be plotted without lifting the pen from the paper" —which is the intuitive notion of a continuous function.[3] The formal definition of continuity is the following:

Definition 1.5.1 (Continuity) *A real function f is said to be* ***continuous***

3: We say 'mainly' because there are very weird functions, which one would intuitively not refer to them as continuous, and nevertheless they are continuous in some subsets. But we shall not be concerned with these functions in this course. We will rather focus on practical, "sensible" functions.

at a point a of its domain if

$$\lim_{x \to a} f(x) = f(a). \tag{1.12}$$

Continuous functions are very nice, and so if you sum/multiply/divide two continuous functions, you get a continuous function. The composition of two continuous functions is continuous, and the inverse of a continuous functions is continuous too.

Finally, two important properties of continuous functions. A continuous function in a closed interval reaches its maximum and minimum values within the interval (in particular, it is bounded). It also reaches all the intermediate values between the maximum and the minimum.

Discontinuities

Discontinuities are points where a function is not continuous. There are several reasons why a function may not be continuous at a point, and some of them bear a specific name.

A function like $f(x) = \dfrac{\sin x}{x}$ is continuous in all $\mathbb{R}$ except $x = 0$, because the denominator vanishes at that point. However, the function has a well defined limit at that point (try to guess it with the calculator):

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

So we can re-define the function to be

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1 & x = 0, \end{cases}$$

and now it is continuous everywhere in $\mathbb{R}$. One such discontinuity is called an **avoidable discontinuity** because it can be "avoided" by properly defining the function.

The case of the Heaviside step function

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0, \end{cases}$$

typifies a stronger case of discontinuity, which cannot be avoided. The function is continuous in $\mathbb{R} - \{0\}$ (because it is a constant for $x < 0$ and for $x > 0$), but at $x = 0$ the left-handed limit is 0 whereas the right-handed limit is 1. So the limit when $x \to 0$ does not exist because, although the two one-sided limits exist, they are different. This is a **jump discontinuity** because the graph of the function "jumps" at that point.

In some cases the function is not continuous because the one or both of the two one-sided limits is $\pm\infty$. Such is the case of $1/x$ or $\log x$. We say that the function has a **singularity** at that point. We also call it an **asymptote**.

Finally, a function can be discontinuous simply because it has no limit at a point. For instance, $\sin\frac{1}{x}$ is continuous in $\mathbb{R} - \{0\}$ because the limit when $x \longrightarrow 0$ does not exist.

Proposed Exercise 1.5.1 Which kind of discontinuity has the function $f(x) = x\sin\frac{1}{x}$ at $x = 0$?

Exercises

Exercise 1.1 Determine the domain of the following functions:

(i) $f(x) = \dfrac{1}{x^2 - 5x + 6}$;

(ii) $f(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$;

(iii) $f(x) = \dfrac{1}{x - \sqrt{1 - x^2}}$;

(iv) $f(x) = \sqrt{1 - \sqrt{4 - x^2}}$;

(v) $f(x) = \dfrac{1}{1 - \log x}$;

(vi) $f(x) = \log(x - x^2)$;

(vii) $f(x) = \dfrac{\sqrt{5 - x}}{\log x}$;

Exercise 1.2

(a) If f and g are both odd functions, what are $f + g$, fg, and $f \circ g$?
(b) And what are the same functions if now f is even and g is odd?

Exercise 1.3 Check whether the following functions are even or odd:

(i) $f(x) = \dfrac{x}{x^2 + 1}$;

(ii) $f(x) = \dfrac{x^2 - x}{x^2 + 1}$;

(iii) $f(x) = \dfrac{\sin x}{x}$;

(iv) $f(x) = \cos(x^3)\sin(x^2)e^{-x^4}$;

(v) $f(x) = \dfrac{1}{\sqrt{x^2 + 1} - x}$;

(vi) $f(x) = \log\left(\sqrt{x^2 + 1} - x\right)$.

Exercise 1.4

(a) Determine which of these functions are injective. For those that are obtain their inverse. For those that are not, find two points with the same image.

 (i) $f(x) = 7x - 4$;

 (ii) $f(x) = \sin(7x - 4)$;

 (iii) $f(x) = (x + 1)^3 + 2$;

 (iv) $f(x) = \dfrac{x + 2}{x + 1}$;

 (v) $f(x) = x^2 - 3x + 2$;

 (vi) $f(x) = \dfrac{x}{x^2 + 1}$;

 (vii) $f(x) = e^{-x}$;

 (viii) $f(x) = \log(x + 1)$.

(b) Prove that $f(x) = x^2 - 3x + 2$ is injective in $(3/2, \infty)$.
(c) Determine if those same functions are surjective and bijective in their domains.

Exercise 1.5 Consider the function $f(x) = 3\sin(2x - \pi) + 1$.

1. Determine the amplitude, defined as $A = \max f(x) - \min f(x)$.
2. Determine the period, defined as the minimal value c that yields $f(x + c) = f(x)$ for all $x \in \mathbb{R}$.
3. Determine the phase shift with respect to $\sin x$.
4. Determine the vertical shift with respect to $\sin x$.

Exercise 1.6 Use the formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

to prove the following identities:

1. $\sin(x + \pi/2) = \cos x$.
2. $\cos(x - \pi/2) = \sin x$. (what does this tell you about $\sin x$ and $\cos x$?)
3. $1 = \cos^2 x + \sin^2 x$.
4. $\cos^2 x = \frac{1+\cos 2x}{2}$.
5. $\sin^2 x = \frac{1-\sin 2x}{2}$.

Exercise 1.7 Find all solutions for x in the interval $[0, 2\pi)$ for the equation $2\cos x - 3 = 0$.

Exercise 1.8

(a) Describe the function g in terms of f in the following cases ($c \in \mathbb{R}$ is a constant):

(i) $g(x) = f(x) + c$;
(ii) $g(x) = f(x + c)$;
(iii) $g(x) = f(cx)$;
(iv) $g(x) = f(1/x)$;
(v) $g(x) = f(|x|)$;
(vi) $g(x) = |f(x)|$;
(vii) $g(x) = 1/f(x)$;
(viii) $g(x) = \max\{f(x), 0\}$.

(b) Plot the functions when $f(x) = x^2$.
(c) Plot the functions when $f(x) = \sin x$.

Exercise 1.9 Sketch, using the fewest possible calculations, the graph of the following functions:

(i) $f(x) = (x + 2)^2 - 1$;
(ii) $f(x) = \sqrt{4 - x}$;
(iii) $f(x) = x^2 + \dfrac{1}{x}$;
(iv) $f(x) = \dfrac{1}{1 + x^2}$;
(v) $f(x) = \min\{x, x^2\}$;
(vi) $f(x) = |e^x - 1|$;
(vii) $f(x) = |x^2 - 1|$;
(viii) $f(x) = 1 - e^{-x}$;
(ix) $f(x) = \log(x^2 - 1)$;
(x) $f(x) = x\sin(1/x)$.

Exercise 1.10 Calculate the following limits, simplifying the common factors that numerator and denominator may contain:

(i) $\displaystyle\lim_{x\to a} \frac{x^n - a^n}{x - a}$, $n \in \mathbb{N}$;

(ii) $\displaystyle\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$;

(iii) $\displaystyle\lim_{x\to 0} \frac{1 - \sqrt{1 - x^2}}{x^2}$;

(iv) $\displaystyle\lim_{x\to 1} \left(\frac{1}{\sqrt{x} - 1} - \frac{2}{x - 1}\right)$.

HINT: This formula will be useful:

$$x^n - y^n = (x-y)\sum_{k=1}^{n} x^{n-k}y^{k-1} = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

Exercise 1.11 Calculate the following limits:

(i) $\displaystyle\lim_{x\to\infty} \frac{x^3 + 4x - 7}{7x^2 - \sqrt{2x^6 + x^5}}$;

(ii) $\displaystyle\lim_{x\to\infty} \frac{x + \sin x^3}{5x + 6}$;

(iii) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$;

(iv) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right)$;

Exercise 1.12 Study the continuity of the following functions:

(i) $f(x) = \dfrac{e^{-5x}+\cos x}{x^2-8x+12}$;

(ii) $f(x) = e^{3/x} + x^3 - 9$;

(iii) $f(x) = x^3 \tan(3x+2)$;

(iv) $f(x) = \begin{cases} \sin(\pi x), & x < -1, \\ |x| - x, & -1 \le x < 1, \\ (x-1)^3, & x \ge 1; \end{cases}$

(v) $f(x) = \begin{cases} x^2, & x \le -2, \\ |x^2-1|, & -2 < x < 2, \\ 4x-5, & x \ge 2; \end{cases}$

(vi) $f(x) = \begin{cases} (x-1)^2, & x > 1, \\ x - \lfloor x \rfloor, & -1 \le x \le 1, \\ x+1, & x < -1. \end{cases}$

Exercise 1.13 Bolzano's theorem states that a continuous function in $[a,b]$ where the sign of $f(a)$ and $f(b)$ is different has to cross zero. Which of these equations have at least one solution ($f(x) = 0$) in the specified set?

(i) $x^2 - 18x + 2 = 0$, in $[-1, 1]$;

(ii) $x - \sin x = 1$, in $\mathbb{R}$;

(iii) $e^x + 1 = 0$, in $\mathbb{R}$;

(iv) $\cos x + 2 = 0$, in $\mathbb{R}$;

2 Derivatives

2.1 Concept and definition

Derivatives are introduced to characterise the *rate of variation* of a function with a number. The rate of variation measures how much the function $f(x)$ increases (positive) or decreases (negative) per unit of variation of the variable x. Thus, within the interval $[a, x]$ this rate will be

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}.$$

Figure 2.1 illustrates that the narrower the interval $[a, x]$ where the variation is measured the more accurate the estimated rate[1]. Ideally, the measure would be perfect if this interval were infinitely narrow. This is the notion of *derivative* and the motivation of its definition:

1: Think of measuring the speed of a car by dividing the distance it has run in a given time.

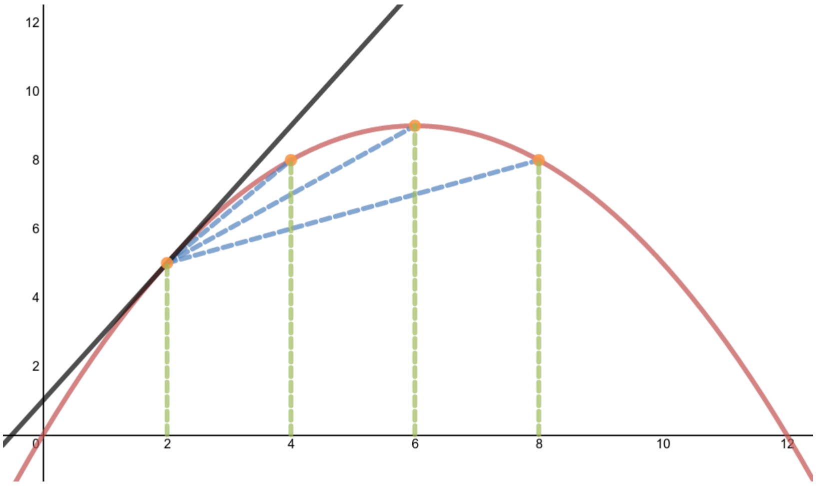


Figure 2.1: The rate of variation of $f(x)$ as obtained for different intervals.

Definition 2.1.1 (Derivative) *The **derivative** of the function f at the point a of its domain is defined as*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \tag{2.1}$$

provided the limit exists. (When it does, we say that the function is differentiable *at a.)*

Figure 2.1 also shows that $f'(a)$ —the rate of variation of $f(x)$ at $x = a$— coincides with the *slope* of the straight line *tangent* to the graph of $f(x)$ at the point $(a, f(a))$ —which is an important geometric characterization of the derivative concept.

Remark 2.1.1 Often you will see the derivative denoted as

$$f'(a) = \frac{df}{dx}(a).$$

This is Leibniz's notation —a bit more mnemotechnical because it reminds that the derivative is, after all, a rate of change of f.

Example 2.1.1 Consider the function $f(x) = x^2$. Its derivative at any point x would be, according to the definition,

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0}(2x + h) = 2x.$$

Therefore $f'(x) = 2x$.

We can generalize this result. Using Newton's binomial formula:

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^n h^{n-k},$$

we can prove that the derivative of $f(x) = x^n$, with $n \in \mathbb{N}$ arbitrary, at any point $x \in \mathbb{R}$ is $f'(x) = nx^{n-1}$. (Note that this formula holds even if $n = 0$, for which $f(x) = 1$.)

Example 2.1.2 Let $f(x) = \sin x$ and $g(x) = \cos x$. By definition

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

But

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = -\lim_{h\to 0} h \frac{1 - \cos h}{h^2} = -0 \cdot \frac{1}{2} = 0,$$

hence $f'(x) = \cos x$.

Similarly

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = -\sin x. \end{aligned}$$

Thus we have the result $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.

Example 2.1.3 Let $f(x) = e^x$ and compute

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} = e^x.$$

We say that f is differentiable in the interval (a, b) if it is differentiable at every point of the interval.

The function f', defined as

$$\begin{aligned} f' : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f'(x), \end{aligned} \tag{2.2}$$

where A is the set of points where f is differentiable, is called the **derivative function** of f (or simply the *derivative* of f).

Likewise, we can introduce higher order derivatives. For instance, f'' is the *second derivative* of f, i.e., the derivative function of f'. Or f''' is the *third derivative* of f, i.e., the derivative function of f''. And so on. (Beyond the third derivative it is customary to denote higher order derivatives as $f^{(n)}$, the nth derivative of f.)

The following theorem emphasises that differentiability is a more restrictive property than continuity.

Theorem 2.1.1 *If f is differentiable at a it is also continuous at a.*

An obvious consequence of this theorem is that discontinuous functions are not differentiable at the discontinuities.

Example 2.1.4 Function $f(x) = |x|$ is continuous in $\mathbb{R}$, however, $f'(0)$ does not exist. The reason is that

$$\lim_{x \to 0^+} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

because $|x| = x$ for $x \geq 0$. However

$$\lim_{x \to 0^-} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

because $|x| = -x$ for $x < 0$. Therefore the limit defining $f'(0)$ does not exists because the left-handed and right-handed limits are different.

2.2 Introduction to Differential Equations

Think of an animal population, and let $N(t)$ be the number of animals at time t.[2] How does N change? Assuming there is no migration, N will increase when there are births, and decrease when there are deaths.

2: Note that until now we have been using y as a function of x. Here the *independent* variable is time.

So, if we look at time $t + \Delta t$, where Δt is a small interval, we can write

$$N(t + \Delta t) = N(t) + bN(t)\Delta t - dN(t)\Delta t, \tag{2.3}$$

where b and d are the number of births and deaths per capita and per unit time, respectively. Note that we have written these rates as constants. This is, of course, a massive simplification: we are assuming that these rates are constant, and this will have consequences for the way the model behaves. There is nothing wrong with this: all models have assumptions, we just need to be clear on what they are. The validity of a model will depend strongly on its assumptions. The assumptions that we are making here are: (1) all animals are capable of giving birth, (2) an animal's ability to give birth is constant over its lifetime from birth to death and (3) all animals have the same likelihood of giving birth. Then for each animal, there is a single constant rate b at which that animal gives birth. And similarly with death: (1) every animal has the same likelihood of dying, (2) the death rate does not depend on the number of animals, (3) the death rate does not vary with time.

Now, it seems natural to think that, as we make Δt smaller, our knowledge of the population size will be better. With some algebra, we can rewrite eq. (2.3) as

$$\frac{N(t+\Delta t)-N(t)}{\Delta t} = (b-d)N(t) \tag{2.4}$$

and, taking the limit when $\Delta t \to 0$, we get

$$N'(t) = rN(t), \tag{2.5}$$

where $r = b - d$ is the effective growth rate. This equation is a **differential equation**, because it involves derivatives. The solution of a differential equation is not a number, but a function. In this case, the function $N(t)$ that we ignore. Note that in this case, as in many real-life problems, it is quite easy for us to understand how a variable changes in time (or space), but not so easy to know its value at a given point in time (or space). Hence, differential equations are very useful.

Proposed Exercise 2.2.1 We will see how to solve (some) differential equations later in the course, but see if you can find a solution of the form $N(t) = e^{at}$, by finding $N'(t)$ and substituting into Equation 2.5. What is the value of a? Is that solution unique?

Actually, for many real-life problems we will not know how to solve the differential equation analytically[3] but we will be able to solve it numerically. We will not cover numerical simulations in this course, but know that this, too, is a powerful tool for modelling.

3: That means finding a function $N(t)$ in terms of elementary functions.

2.3 Algebraic properties of derivatives

The fact that derivatives are defined as limits leads to the following algebraic properties:

Proposition 2.3.1 *Let f and g be two differentiable functions (in an appropriate set). Then:*

(i) $(\lambda f + \mu g)' = \lambda f' + \mu g'$, *where* $\lambda, \mu \in \mathbb{R}$; (linearity)

(ii) $(fg)' = f'g + fg'$; (Leibniz's rule)

(iii) $(f \circ g)' = (f' \circ g)g'$; (chain rule)

(iv) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, *provided* $g \neq 0$; (quotient rule)

(v) $\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}$; (inverse rule)

The following examples illustrate how these rules can be applied to obtain new derivatives:

Example 2.3.1 If $f(x) = e^x$ then $f^{-1}(x) = \log x$. Therefore

$$(\log x)' = \frac{1}{e^{\log x}} = \frac{1}{x}.$$

Logarithms can have a different base, say $a > 0$. They are denoted $\log_a x$ and form the inverse function of a^x. Now $a^x = e^{x \log a}$, so by the chain rule

$$(a^x)' = \left(e^{x\log a}\right)' = \left(e^{x\log a}\right)\log a = a^x \log a.$$

Therefore

$$(\log_a x)' = \frac{1}{a^{\log_a x}\log a} = \frac{1}{x\log a}.$$

Example 2.3.2 Function $f(x) = x^\alpha$, with $\alpha \in \mathbb{R}$, can be written as $f(x) = e^{\alpha \log x}$. Thus, applying the chain rule,

$$(x^\alpha)' = e^{\alpha\log x}\frac{\alpha}{x} = x^\alpha\frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

Example 2.3.3 If $f(x) = \sin x$ in $[-\pi/2, \pi/2]$ then $f^{-1}(x) = \arcsin x$. Thus,

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}.$$

But $\cos x = \sqrt{1-\sin^2 x}$ in $[-\pi/2, \pi/2]$, so

$$(\arcsin x)' = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}},$$

because $\sin(\arcsin x) = x$.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	$\sin x$	$\cos x$
x^α	$\alpha x^{\alpha-1}$	$\cos x$	$-\sin x$
e^x	e^x	$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$
a^x	$a^x \log a$	$\arctan x$	$\frac{1}{1+x^2}$
$\log x$	$\frac{1}{x}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\log_a x$	$\frac{1}{x\log a}$	$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$

Table 2.1: Derivatives of most elementary functions. Here $c, \alpha \in \mathbb{R}$, $a > 0$.

Exercises

Exercise 2.1 Let f and g be differentiable functions in $\mathbb{R}$. Write down the derivative of the following functions in their respective domains:

(i) $h(x) = \sqrt{f(x)^2 + g(x)^2}$;

(ii) $h(x) = \arctan\left(\dfrac{f(x)}{g(x)}\right)$;

(iii) $h(x) = f\big(g(x)\big)e^{f(x)}$;

(iv) $h(x) = \log\big(g(x)\sin f(x)\big)$;

(v) $h(x) = f(x)^{g(x)}$;

(vi) $h(x) = \dfrac{1}{\log\big(f(x) + g(x)^2\big)}$.

Exercise 2.2 Check that the following functions satisfy the specified differential equations, where c, c_1, and c_2 are constants:

(i) $f(x) = \dfrac{c}{x}$ satisfies $xf' + f = 0$;

(ii) $f(x) = x\tan x$ satisfies $xf' - f - f^2 = x^2$;

(iii) $f(x) = c_1 \sin 3x + c_2 \cos 3x$ satisfies $f'' + 9f = 0$;

(iv) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$ satisfies $f'' - 9f = 0$;

(v) $f(x) = c_1 e^{2x} + c_2 e^{5x}$ satisfies $f'' - 7f' + 10f = 0$;

(vi) $f(x) = \log(c_1 e^x + e^{-x}) + c_2$ satisfies $f'' + (f')^2 = 1$.

Exercise 2.3 Prove the identities (valid only in the specified regions)

(i) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$, for $x > 0$;

(ii) $\arctan \dfrac{1+x}{1-x} - \arctan x = \dfrac{\pi}{4}$, for $x < 1$;

(iii) $2\arctan x + \arcsin \dfrac{2x}{1+x^2} = \pi$, for $x \geq 1$.

HINT: Differentiate the equation and check one point of the specified region.

Exercise 2.4 At which points does the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent?

Exercise 2.5 Given the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0, \\ 0 & x = 0, \end{cases}$$

calculate the angle between the tangents on the left and on the right of its graph at $x = 0$.

Exercise 2.6 Find the sets where the function $f(x) = \sqrt{x+2}\arccos(x+2)$ is continuous and differentiable.

Exercise 2.7 Calculate the smallest α for which $f(x) = |\alpha x^2 - x + 3|$ is differentiable in $\mathbb{R}$.

Exercise 2.8 Given the function

$$f(x) = \begin{cases} a + bx^2, & |x| \leq c, \\ \dfrac{1}{|x|}, & |x| > c, \end{cases} \qquad c > 0,$$

find a and b so that it is continuous and differentiable in $\mathbb{R}$.

Exercise 2.9 Determine the sets where it is continuous and where it is differentiable

$$f(x) = \begin{cases} \dfrac{3 - x^2}{2}, & x < 1, \\ \dfrac{1}{x}, & x \geq 1, \end{cases}$$

Taylor Expansions 3

3.1 Taylor Polynomial

From our definition of derivative as the limit of a quotient, we can write

$$\frac{f(x+h)-f(x)}{h} \approx f'(x) \iff \frac{f(x)-f(a)}{x-a} \approx f'(a),$$

or, in another words,

$$f(x) \approx f(a) + f'(a)(x-a).$$

This approximation will be better as $x \to a$ and will be exact in the limit.

Note that both $f(a)$ and $f'(a)$ are constants, and so we are approximating f by a polynomial of degree 1, a line.

Take $f(x) = e^x$ and $a = 0$. What is the linear approximation of e^x at 0?

The derivative of e^x is itself e^x which at $x = 0$ is equal to 1. So $e^x \approx 1 + x$.

How good is the approximation? Take $x = 0.1$ for instance. The true value is $e^{0.1} = 1.10517091808\ldots$, so the error appears in the third decimal. In general, the error will be of the same order of magnitude as x^2[1] , which means it will be some constant E_2 times x^2.

1: Why this is so is harder to explain, but trust me

But we know that e^x is not linear. Can we increase the degree of the polynomial so that the approximation is better? In other words: if we write as approximation $1 + x + c_2x^2$, what is the value that c_2 needs to take so that the approximation is good enough? Note that the error will now be of the order of magnitude of x^3, something like E_3x^3, with E_3 a constant[2] :

$$e^x = 1 + x + c_2x^2 + E_3x^3.$$

2: Again, trust me. It would take us too long to explain this.

Now, if we differentiate twice, we get

$$e^x = 2c_2 + 6E_3x.$$

At $x = 0$ the equation becomes $c_2 = 1/2$. Note that, for a general function f, this will be equal to $f''(0)/2$. So $1 + x + x^2/2$ is the polynomial of second degree that best approximates e^x. And if we substitute $x = 0.1$ like before, and we get 1.105, and the error is just 0.00017, of the same order as $0.1^3 = 0.001$[3]

3: It is actually six times smaller. We'll get there.

We can keep adding terms. If we try to find the polynomial of third degree that best approximates e^x, we have

$$e^x = 1 + x + \frac{x^2}{2} + c_3x^3 + E_4x^4,$$

and differentiating three times we get

$$e^x = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2E_4 x.$$

Again, at $x = 0$ this becomes $c_3 = 1/6$ and, for a general f, it is $c_3 = f'''(0)/3!$. The approximation now is $1.1051666666\ldots$ and the error is only 0.0000042, or around $0.1^4/24$.

In general, we have

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Since we know that the remaining terms will be of the order of x^{n+1} or smaller, we sometimes write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

to mean that this approximation is exact up to terms that vanish faster than x^n.[4]

4: This notation is called Landau's "small o", in case you want to look it up.

This pattern that we have seen for the exponential holds for all functions that are differentiable in a given interval, which leads to the following definition:

Definition 3.1.1 *(Taylor polynomial)The polynomial of nth degree that best approximates the function f at the point a is*

$$P_{n,a}(x) \equiv f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad (3.1)$$

which we will refer to as the nth order ***Taylor polynomial*** *of function f at the point a.*

The error (also called the remainder) of the approximation is the difference between the Taylor polynomial and the function, and is given by

$$R_{n,a}(x) = f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}, \quad (3.2)$$

where $c \in (a, x)$.

Example 3.1.1 Consider the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$. Then $f(0) = 1$ and

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1}, & f'(0) &= \alpha, \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, & f''(0) &= \alpha(\alpha-1), \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, & f'''(0) &= \alpha(\alpha-1)(\alpha-2), \\
&\vdots & &\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}, & f^{(n)}(0) &= \alpha(\alpha-1)\cdots(\alpha-n+1).
\end{aligned}$$

Therefore

$$\begin{aligned}(1+x)^\alpha =& 1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\\ &+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+o(x^n) \quad (x\longrightarrow 0).\end{aligned}$$

There is an interesting notation for this expression derived from the formula for the binomial coefficients. If $\alpha \in \mathbb{N}$,

$$\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Since this formula is meaningful even if $\alpha \in \mathbb{R}$, we use it as a definition and thus write

$$(1+x)^\alpha=\sum_{k=0}^{n}\binom{\alpha}{k}x^k+o(x^n) \quad (x\longrightarrow 0).$$

This is the famous binomial formula as it was first obtained by Newton in 1665.

Proposed Exercise 3.1.1 Find the Taylor polynomial of degree n and centered at $x=0$ for $\sin x$, $\cos x$ and $\log(1+x)$?

Table 3.1: Taylor polynomials of some elementary functions as $x \longrightarrow 0$. (Here $\alpha \in \mathbb{R}$.)

$f(x)$	$P_{k,0}(x)$
$(1+x)^\alpha$	$1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$
$\log(1+x)$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n+1}\frac{x^n}{n}$
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots+\frac{x^n}{n!}$
$\sin x$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
$\cos x$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}$

3.2 Taylor series

Take the function $f(x)=(1-x)^{-1}$ and start calculating its Taylor polynomial at $x=0$. The first derivatives are

$$\begin{aligned}f'(x)&=(1-x)^{-2} &&\Longrightarrow f'(0)=1,\\ f''(x)&=2(1-x)^{-3} &&\Longrightarrow f''(0)=2,\\ f'''(x)&=3!(1-x)^{-4} &&\Longrightarrow f'''(0)=3!,\\ f^{iv}(x)&=4!(1-x)^{-5} &&\Longrightarrow f^{iv}(0)=4!,\\ &\vdots\end{aligned}$$

There seems to be a pattern, that is, $f^{(n)} = n!$. If we substitute this into the formula for the Taylor polynomial of degree n centered at $x = 0$, we get:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Now, we can keep doing this forever, and as $n \longrightarrow \infty$ the error term will go to zero: since f is infinitely differentiable in an interval around $x = 0$[5] then the $n+1$-th derivative, $f^{(n+1)}(c)$ will be bounded by some real number M. But $(n+1)! \longrightarrow \infty$ as $n \longrightarrow infty$, so $E_{n,0}(x) \longrightarrow 0$.

This means that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \tag{3.3}$$

where the right-hand side denotes an infite sum, which in mathematical terms is called a **series**. Because this sum is made up of powers of x, we call it a **power series**. There are many types of power series[6] but, because this comes from the Taylos polynomial, we call it the **Taylor series**.

What we are saying here is that our function $f(x) = (1-x)^{-1}$ is exactly equal to its Taylor series in a given interval, in this case $(-1, 1)$. So this means that we can understand any smooth function as an infinite polynomial, and that we can get any information about the function we need from this series.

Proposed Exercise 3.2.1 Can you do the same for e^x, $\sin x$, $\cos x$ and $\log(1+x)$?

Ok, so the function is identical to its Taylor series, but is this true for all x in the domain? If you substitute $x = 2$ in Equation (3.3), and keep adding terms, you'll see that the sum does not approach or converge (in mathematical terms) to the actual value of the function $1/(1-2) = -1$[7] . When does this happen?

We don't have time to explain why, but a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ will converge if

$$\lim_{n\to\infty} \sqrt[n]{|a_n||x-a|^n} < 1 \qquad \Leftrightarrow \qquad \left(\lim_{n\to\infty} \sqrt[n]{|a_n|}\right)|x-a| < 1. \tag{3.4}$$

Definition 3.2.1 *(Convergence radius)We can define the number $\rho > 0$ by the formula*

$$\frac{1}{\rho} \equiv \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \tag{3.5}$$

*We refer to ρ as the **convergence radius** of the series because condition* (3.4) *holds for every x such that*

$$|x-a| < \rho. \tag{3.6}$$

In other words, the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely in the interval $(a-\rho, a+\rho)$.

5: More precisely, this is true for this example in ther interval $(-1, 1)$. The reasons for this are hard to explain without some knowledge of complex variable.

6: In general, a power series is a series of the form $\sum_{n=0}^{\infty} a_n(x-a)^n, \quad a_n \in \mathbb{R}$.

7: Especially since every summand in the series is positive!!

In summary, the power series converges if $x \in (a-\rho, a+\rho)$ and diverges (goes to ∞) otherwise, except maybe at $x = a \pm \rho$. At these two points the analysis has to be done on a case-by-case basis.

Proposed Exercise 3.2.2 Find the radius of convergence and show that the interval of convergence of the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

is $(-1, 1)$. What happens at $x = 1$ and $x = -1$?

Example 3.2.1 Consider the Taylor expansion of $f(x) = e^x$ with remainder. Given that $f^{(n)}(x) = e^x$ we will have

$$e^x = \sum_{k=0}^{n} \frac{x^n}{n!} + R_{n,0}(x), \qquad R_{n,0}(x) = e^{\theta x} \frac{x^{n+1}}{(n+1)!}, \qquad 0 < \theta < 1.$$

Since the exponential is an increasing function, $e^{\theta x} < \max\{1, e^x\}$ —that includes the cases $x > 0$ and $x < 0$. Therefore

$$0 < R_{n,0}(x) < \max\{1, e^x\} \frac{x^n}{n!} \to n\infty 0$$

for any given $x \in \mathbb{R}$. Hence

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}.$$

3.3 Numerical approximations

With the expression of the remainder we can find bounds to the error that we incur when approximating a function by its Taylor polynomial of a certain degree. This allows us to obtain numerical values of transcendental functions —which would otherwise be difficult to obtain. Some examples illustrate the method.

Example 3.3.1 We know that

$$\sin x = x - \frac{x^3}{6} + R_{4,0}(x), \qquad R_{4,0}(x) = \frac{\cos(\theta x)}{120} x^5, \qquad 0 < \theta < 1.$$

We of course ignore the value of θ (otherwise $\sin x$ could be exactly computed), but we know that irrespective of θ and x, $|\cos(\theta x)| \leq 1$. Thus,

$$|R_{4,0}(x)| \leq \frac{|x|^5}{120}.$$

Suppose we want to compute $\sin(0.1)$. From the previous inequality $|R_{4,0}(x)| \leq 8.3333 \times 10^{-8}$. Now compare:

$$\sin(0.1) = \mathbf{0.09983}341664\ldots, \qquad P_{4,0}(0.1) = \mathbf{0.09983}333333\ldots$$

The error incurred using this simple approximation is 8.3313×10^{-8}, very close to our estimate.

Suppose we do not want our error to be larger than 10^{-5}. What is the largest x for which we can use this approximation? To answer this question we simply set the estimate to the error tolerance and find $|x|$:

$$\frac{|x|^5}{120} = 10^{-5} \qquad \Rightarrow \qquad |x| = \sqrt[5]{120} \times 0.1 \approx 0.26.$$

Example 3.3.2 Imagine that we want to compute $\sqrt{3.8}$. We can do it by expanding the function $\sqrt{4-x}$ around $x = 0$. Thus,

$$\begin{aligned} f(x) &= \sqrt{4-x}, & f(0) &= \sqrt{4} = 2, \\ f'(x) &= \frac{-1}{2\sqrt{4-x}}, & f'(0) &= \frac{-1}{2\sqrt{4}} = -\frac{1}{4}, \\ f''(x) &= \frac{-1}{4(4-x)^{3/2}}, & f''(0) &= \frac{-1}{4 \cdot 4^{3/2}} = -\frac{1}{32}, \\ f'''(x) &= \frac{-3}{8(4-x)^{5/2}}. \end{aligned}$$

Then

$$\sqrt{4-x} = 2 - \frac{x}{4} - \frac{x^2}{64} + R_{2,0}(x),$$

where

$$R_{2,0}(x) = \frac{-1}{16(4-\theta x)^{5/2}} x^3, \quad 0 < \theta < 1.$$

If $x > 0$,

$$|R_{2,0}(x)| < \frac{x^3}{16(4-x)^{5/2}} = \frac{x^3}{16\left(\sqrt{4-x}\right)^5}.$$

Now we can estimate

$$\sqrt{3.8} = P_{2,0}(0.2) = 2 - \frac{0.2}{4} - \frac{(0.2)^2}{64} = 1.949375\ldots$$

and use this estimation in the error bound

$$|R_{2,0}(x)| < \frac{(0.2)^3}{16(1.949375\ldots)^5} \approx 1.78 \times 10^{-5}.$$

As a matter of fact,

$$\sqrt{3.8} = \mathbf{1.949}358869\ldots, \qquad P_{2,0}(0.2) = \mathbf{1.949}375,$$

the difference being 1.61×10^{-5}.

Exercises

Exercise 3.1 Write the Taylor polynomial $P_{5,0}(x)$ for these functions:

(i) $e^x \sin x$;
(ii) $e^{-x^2} \cos 2x$;
(iii) $\sin x \cos 2x$;
(iv) $e^x \log(1-x)$;
(v) $\sin^2 x$;
(vi) $\dfrac{1}{1-x^3}$.

Exercise 3.2 Write the polynomial $x^4 - 5x^3 + x^2 - 3x + 4$ in powers of $x - 4$.

HINT: Note that, if f is a polynomial of degree n, the nth degree Taylor polynomial of f centered at a is exactly identical to f.

Exercise 3.3 Write the Taylor polynomial $P_{n,a}(x)$ for these functions around the specified a:

(i) $f(x) = 1/x$ around $a = -1$;
(ii) $f(x) = xe^{-2x}$ around $a = 0$;
(iii) $f(x) = (1+e^x)^2$ around $a = 0$;
(iv) $f(x) = \sin x$ around $a = \pi$.

Exercise 3.4 Use a Taylor polynomial of the specified degree to provide an approximation to these numbers, and give an upper bound for the error incurred:

(i) $\dfrac{1}{\sqrt{1.1}}$, degree 3;
(ii) $\sqrt[3]{28}$, degree 2.
(iii) $\log(3/2)$, degree 4.

Exercise 3.5 Given the function $f(x) = \cos x + e^x$,

(i) find its Taylor polynomial $P_{3,0}(x)$;
(ii) estimate an upper bound for the error incurred if $-1/4 \leq x \leq 1/4$.

Exercise 3.6 What is the smallest degree Taylor polynomial necessary to approximate the function $f(x) = e^x$ in $[-1, 1]$ with at least three exact decimal places?

Exercise 3.7 Expand in power series the following functions, specifying the domain of validity of those expansions:

(i) $f(x) = \sin^2 x$;
(ii) $f(x) = \log \sqrt{\dfrac{1+x}{1-x}}$;
(iii) $f(x) = \dfrac{x}{a+bx}$;
(iv) $f(x) = \dfrac{1}{2-x^2}$;

4 Local Behavior of Functions

4.1 Local Extrema

We will see here a set of results related to the local behaviour of a function (i.e., the behaviour within intervals). To begin with, we need to define local maxima and minima.

We say that a function f has a **local maximum** at a point a of its domain, if there is some interval $(a-\delta, a+\delta)$ such that $f(x) \leq f(a)$ for all $x \in (a-\delta, a+\delta)$.

We say that a function f has a **local minimum** at a point a of its domain, if there is some interval $(a-\delta, a+\delta)$ such that $f(x) \geq f(a)$ for all $x \in (a-\delta, a+\delta)$.

Local maxima and minima are collectively called **local extrema.** If local extrema remain extrema for all x in the domain of f, they are **absolute extrema.**

Theorem 4.1.1 (Derivatives at local extrema) *If f has a local extremum at a point a where it is differentible then $f'(a) = 0$.*

However:

Example 4.1.1 Consider the function $f(x) = |x(1-x)|$. We know that $x(1-x) \geq 0$ if $0 \leq x \leq 1$, and $x(1-x) < 0$ if $x < 0$ or $x > 1$. Then we can rewrite

$$f(x) = \begin{cases} x(1-x), & 0 \leq x \leq 1, \\ x(x-1), & x < 0 \text{ or } x > 1. \end{cases}$$

Let us compute the derivative,

$$f'(x) = \begin{cases} 1-2x, & 0 < x < 1, \\ 2x-1, & x < 0 \text{ or } x > 1. \end{cases}$$

The derivative at $x = 0$ and $x = 1$ does not exists because, being $f(0) = 0$ and $f(x) = x(x-1)$ for $x < 0$,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x(x-1)}{x} = \lim_{x \to 0^-} (x-1) = -1.$$

However, since $f(x) = x(1-x)$ for $x > 0$,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x(1-x)}{x} = \lim_{x \to 0^+} (1-x) = 1.$$

Since both one-sided limits are different the limit does not exist. For $x = 1$ the argument is similar.

Now to find the local extrema we need to look for the solutions of

$f'(x) = 0$. This equation boils down to $2x = 1$, whose solution is $x = \frac{1}{2}$.

Figure 4.1 presents a plot of $f(x)$. One can clearly see that $x = \frac{1}{2}$ is indeed a local maximum —albeit not absolute, because there are points where $f(x) > f(1/2)$—; however, we can also see that $x = 0$ and $x = 1$ are local minima, but they are not contained in the equation $f'(x) = 0$. (Incidentally, these minima are both absolute.)

There is no contradiction with the theorem though, because, as we have just seen, the function is not differentiable at those points —a premise of the theorem.

This example brings about the point that, when looking for extrema, we need to check not only the solutions of $f'(x) = 0$, but also the points where $f'(x)$ does not exist.

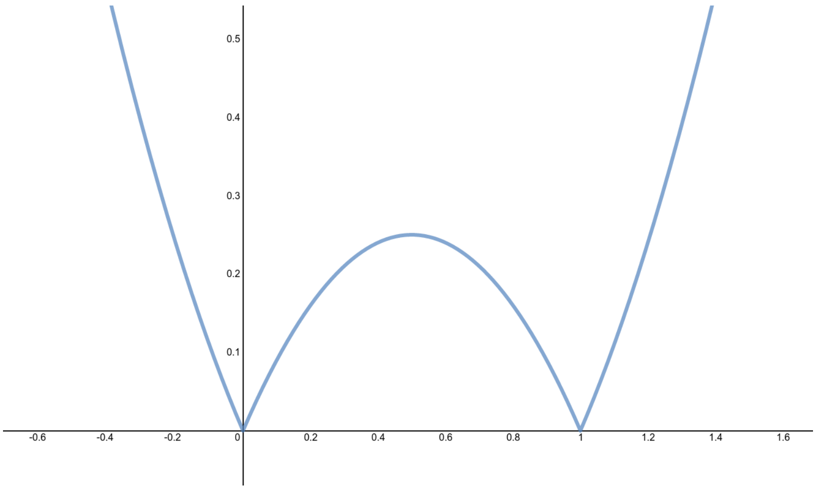


Figure 4.1: Plot of the function $f(x) = |x(1-x)|$.

Remark 4.1.1 Notice also that $f'(c) = 0$ does not imply that c is an extremum. For instance take $f(x) = x^3$. Clearly $f'(0) = 0$, however there is no extremum at $x = 0$ because $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$. We will see later how to characterise maxima and minima using higher-order derivatives.

From the zeroth-degree Taylor polynomial, we know that $f(b) - f(a) = f'(c)(b-a)$ where $c \in (a, b)$[1] . From this we can conclude:

1: This result is also known as the **mean value theorem.**

Corollary 4.1.2 *(i) If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant in (a, b).*
(ii) If $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = g(x) + k$ in (a, b), with $k \in \mathbb{R}$ a constant.
(iii) If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing in (a, b).
(iv) If $f'(x) < 0$ for all $x \in (a, b)$ then f is strictly decreasing in (a, b).

These resuls are useful in identifying the nature of extrema, as this example illustrates:

Example 4.1.2 Find the absolute extrema of the function $f(x) = 2x^{5/3} + 5x^{2/3}$ in the interval $[-8, 1]$.

There are four steps to solve a problem like this:

(1) Find the set where $f'(x)$ exists, and solve the equation $f'(x) = 0$ within that set.
(2) Take all solutions of $f'(x) = 0$ along with the points where $f'(x)$ does not exist.
(3) Check whether any of those point is a local extremum by checking the sign of f' on their left and on their right.
(4) Compare the value of $f(x)$ in all those points as well as the values at the extremes of the interval. Select the largest and the smallest and identify the absolute extrema.

In the case we are dealing with here

$$f'(x) = \frac{10}{3}(x^{2/3} + x^{-1/3}) = \frac{10}{3}(x+1)x^{-1/3}.$$

This function is well defined for all $x \neq 0$. At $x = 0$ the derivative does not exists because the limit

$$\lim_{x\to 0} \frac{2x^{5/3} + 5x^{2/3}}{x} = \lim_{x\to 0} \left(2x^{2/3} + 5x^{-1/3}\right)$$

diverges.

Now, the solution of $f(x) = 0$ is $x = -1$, and $f'(x) > 0$ for $x < -1$ (notice that $x^{-1/3} < 0$ whenever $x < 0$), but $f'(x) < 0$ for $-1 < x < 0$. The function thus increases on the left of $x = -1$ and decreases on the right, therefore there is a *local maximum* at $x = -1$.

As for $x = 0$, $f'(x) < 0$ for $-1 < x < 0$, but $f'(x) > 0$ for $x > 0$. Thus there is a *local minimum* at $x = 0$.

That is all for local extrema. Concerning absolute extrema we need to compute

$$f(-1) = 3, \qquad f(0) = 0, \qquad f(-8) = -44, \qquad f(1) = 7.$$

So the absolute maximum is at $x = 1$ (the rightmost extreme of the interval) and the absolute minimum is at $x = -8$ (the leftmost extreme of the interval).

Figure 4.2 illustrates what we have just found.

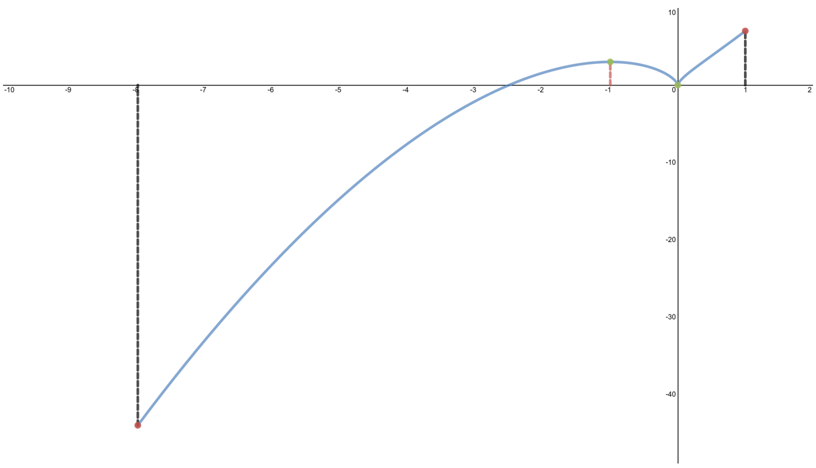


Figure 4.2: Plot of the function $f(x) = 2x^{5/3} + 5x^{2/3}$.

4.2 Convexity and Concavity

We saw in Corollary 4.1.2 that the sign of $f'(x)$ determines wether the function is increasing (positive) or decreasing (negative) at x, and Theorem 4.1.1 showed that at local extrema the function satisfies $f'(x) = 0$ (provided it is differentiable). In its second formulation —with the remainder— Taylor's theorem provides a more detailed information about the local behaviour of a function which has higher order derivatives.

Before getting into it, we need to characterise another qualitative feature of functions: whether their slope increases or decreases. This feature is called *convexity.*

We say that f is **convex** at $x = a$ if it is locally *above* its tangent at that point, i.e., $f(x) > f(a) + f'(a)(x - a)$ for all $0 < |x - a| < \epsilon$, for some $\epsilon > 0$.

Likewise, we say that it is **concave** at $x = a$ if it is locally *below* its tangent a that point, i.e., $f(x) < f(a) + f'(a)(x - a)$ for all $0 < |x - a| < \epsilon$, for some $\epsilon > 0$.

Finally, we say that f has an **inflection point** at $x = a$ if the sign of $f(x) - f(a) - f'(a)(x - a)$ is different for $x < a$ and for $x > a$.

Figure 4.3 illustrates these three behaviours.

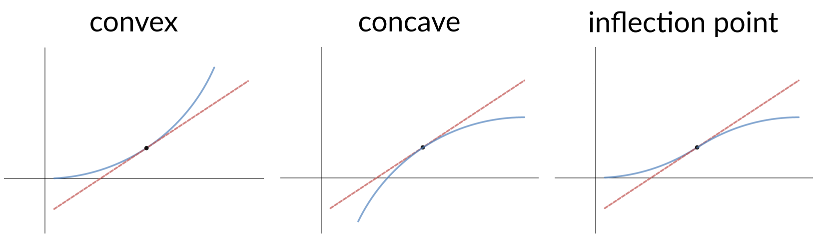


Figure 4.3: Local behaviour of a function with respect to its tangent at a point (convexity).

Suppose that a function f can be differentiated several times (posibly infinitely many) in a certain interval and that the first nonzero derivative beyond the first at $x = a$ is $f^{(n)}(a)$. We can use Taylor's theorem —with Lagrange's remainder— to write

$$f(x) = f(a) + f'(a)(x - a) + \frac{f^{(n)}(c)}{n!}(x - a)^n,$$

where $c = (1 - \theta)a + \theta x$ with $0 < \theta < 1$. One important point to stress here is that, since $f^{(n)}(a) \neq 0$ —so it is either positive or negative—, when x is sufficiently close to a —and so is c— $f^{(n)}(c)$ will have the same sign as $f^{(n)}(a)$. This is key for the argument to come.

Since we can write the Taylor expansion as

$$f(x) - f(a) - f'(a)(x - a) = \frac{f^{(n)}(c)}{n!}(x - a)^n,$$

the sign of the left-hand side —which decides the convexity— will be determined by sign of the product $f^{(n)}(c)(x - a)^n$ or, given what we have just argued, by the sign of the product $f^{(n)}(a)(x - a)^n$.

Now, if n is odd, the sign of $f^{(n)}(a)$ is irrelevant because $(x-a)^n$ has a different sign for $x < a$ and for $x > a$. Therefore a will be an *inflection point.*

If n is even then $(x-a)^n > 0$ for all $x \neq a$. Then the sign is determined by that of $f^{(n)}(a)$. We will then have two possibilities:

(a) $f^{(n)}(a) > 0$, and then the function is *convex*, or
(b) $f^{(n)}(a) < 0$, and then the function is *concave.*

If added to that we have that $f'(a) = 0$, then for n odd nothing changes —hence $x = a$ still is an inflection point—, but for n even the point $x = a$ is a local extremum. A convex extremum ($f^{(n)}(a) > 0$) is a *local minimum* and a concave extremum ($f^{(n)}(a) < 0$) is a *local maximum.*

All these results are summarised in Table 4.1.

n	sign of $f^{(n)}(a)$	$f'(a) \neq 0$	$f'(a) = 0$
odd	$+/-$	inflection point	inflection point
even	$+$	convex	local minimum
even	$-$	convex	local maximum

Table 4.1: Classification of the local behaviour of a function according to the sign of the first nonzero derivative $f^{(n)}(a)$ with $n > 1$.

4.3 Function graphing

All the local information provided by the derivatives can be gathered to sketch a qualitative graph of any function $f(x)$. The steps to follow in graphing a function are these (some of them might not be necessary):

1. **Domain:** Determine precisely the set of points where the function $f(x)$ is defined.
2. **Symmetries:** It is helpful to know whether the function has one of these symmetries:
   (a) *Even:* $f(-x) = f(x)$.
   (b) *Odd:* $f(-x) = -f(x)$.
   (c) *Periodic:* $f(x+c) = f(x)$ for some $c > 0$.

   In the first two cases it is enough to represent the function for $x \geq 0$ (for $x < 0$ it is represented using the symmetry). In the last case it is enough to represent the function in the interval $[0, c]$ (or any other interval of the same lenght) and then reproduce its graph periodically.
   Other symmetries might be possible (e.g., $f(a+x) = \pm f(a-x)$, i.e., f is even/odd around the vertical axis $x = a$).
3. **Continuity and differentiability:** Discontinuities ("jumps") and points where $f'(x)$ does not exists ("cusps") are relevant features of the function, and might be useful in detecting local extrema.
4. **Zeroes:** Finding the solutions of $f(x) = 0$ determines where f crosses the X axis. These points separate regions where the sign of f remains constant.
5. **Growth:** Finding the solutions of $f'(x) = 0$ determines the regions where f increases ($f'(x) > 0$) or decreases ($f'(x) > 0$). Usually this is enough to locate the extrema of f.

6. **Convexity:** The convex/concave regions are usually determined by the sign of $f''(x)$. Inflections points can be inferred from that information (as points where the concavity changes).
7. **Asymptotes:** These are known curves (usually straight lines) which $f(x)$ approaches when it gets close to some points or to $\pm\infty$. The main ones are:

   (a) *Vertical asymptotes:* These are the vertical straight lines through the points $x = a$ where $\lim_{x \to a^{\pm}} f(x) = \pm\infty$.

   (b) *Horizontal asymptotes:* These are the horizontal straight lines $y = \ell$ where ℓ is such that $\lim_{x \to \pm\infty} f(x) = \ell$.

   (c) *Inclined asymptotes:* We say that $y = mx + b$ is an asymptote of $f(x)$ when $x \to \pm\infty$ if

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}, \qquad b = \lim_{x \to \pm\infty} [f(x) - mx].$$

   (In other words, $f(x) = mx + b + o(1)$ $(x \to \pm\infty)$.)

   Other types of asymptote are possible. In general, the curve $y = g(x)$ is an asymptote of f when $x \to \pm\infty$ if $f(x) = g(x) + o(1)$ $(x \to \pm\infty)$.

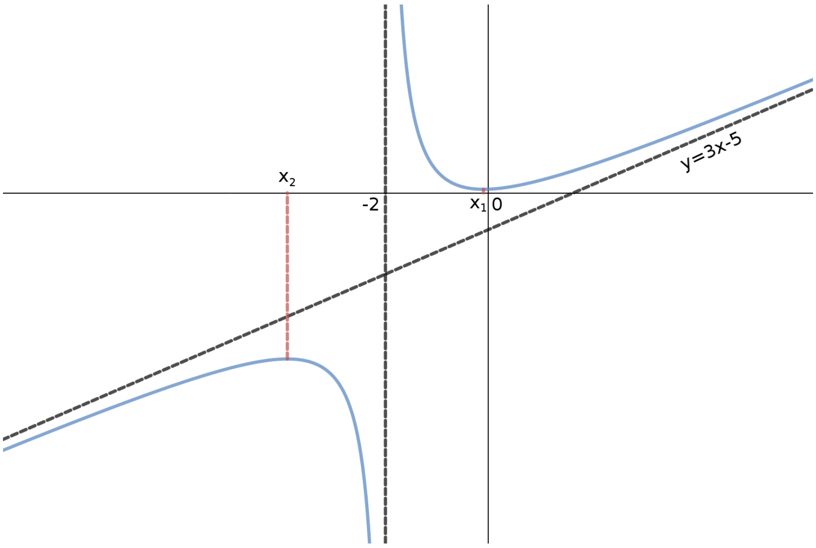


Figure 4.4: Sketch of $f(x) = \dfrac{3x^2 + x + 1}{x + 2}$.

Example 4.3.1 Sketch the graph of

$$f(x) = \frac{3x^2 + x + 1}{x + 2}.$$

The domain of this function is $\mathbb{R} - \{-2\}$ (because the denominator vanishes at that point.) It has no obvious symmetries and, being a rational function, it is continuous and differentiable (an infinite number of times) in all its domain.

We can obtain the derivative as

$$f'(x) = \frac{(6x + 1)(x + 2) - (3x^2 + x + 1)}{(x + 2)^2} = \frac{6x^2 + 13x + 2 - 3x^2 - x - 1}{(x + 2)^2} = \frac{3x^2 + 12x + 1}{(x + 2)^2}.$$

This derivative vanishes when $3x^2 + 12x + 1 = 0$. The roots of this parabola are $x = -2 \pm \sqrt{11/3}$, i.e., $x_1 \approx -0.085$, $x_2 \approx -3.91$. For $x < x_2$ and $x > x_1$ function f increases ($f' > 0$) and for $x_2 < x < x_1$ it decreases ($f' < 0$).

f has no zeros because $3x^2 + x + 1 > 0$ for all $x \in \mathbb{R}$ (the parabola has no roots). So $f(x) < 0$ for $x < -2$ and $f(x) > 0$ for $x > -2$.

It is not necessary to analyse the concavity, as it can be inferred from all the other information, including that of the asymptotes. We know there is a vertical asymptote at $x = -2$ because

$$\lim_{x \to -2^-} f(x) = -\infty, \qquad \lim_{x \to -2^+} f(x) = +\infty.$$

There are no horizontal asymptotes because f diverges when $x \to \pm\infty$. However, we can express the polynomial $P(x) = 3x^2 + x + 1$ in powers of $x + 2$ using Taylor's polynomial, because $P_{2,-2}(x) = P(x)$. As

$$\begin{aligned} P(x) &= 3x^2 + x + 1, & P(-2) &= 11, \\ P'(x) &= 6x + 1, & P'(-2) &= -11, \\ P''(x) &= 6, & P''(-2) &= 6, \end{aligned}$$

we have $P(x) = 11 - 11(x + 2) + 3(x + 2)^2$. Therefore

$$\begin{aligned} f(x) &= \frac{3x^2 + x + 1}{x + 2} = \frac{11 - 11(x + 2) + 3(x + 2)^2}{x + 2} = \frac{11}{x + 2} - 11 + 3(x + 2) = \frac{11}{x + 2} - 5 + 3x \\ &= 3x - 5 + o(1) \quad (x \to \pm\infty), \end{aligned}$$

i.e., $y = 3x - 5$ is an inclined asymptote both when $x \to \pm\infty$.

$f(x)$ is represented in Figure 4.4.

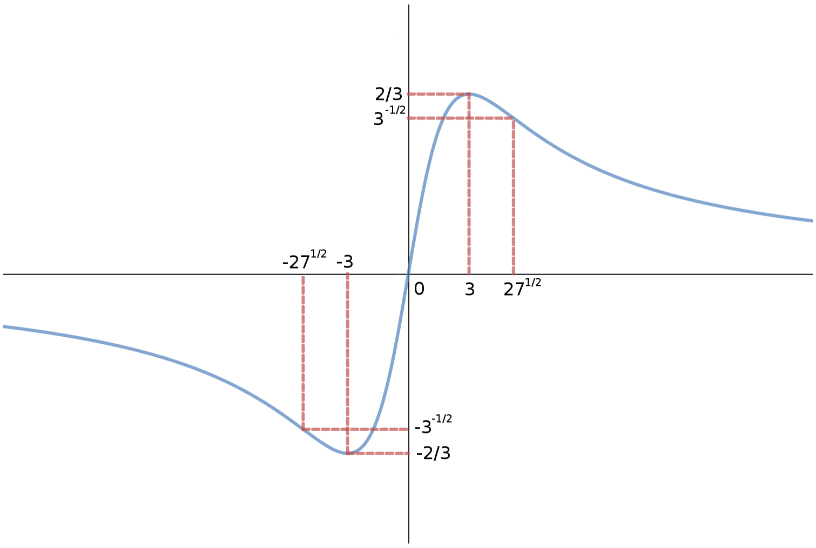


Figure 4.5: Sketch of $f(x) = \dfrac{4x}{x^2 + 9}$.

Example 4.3.2 Sketch the graph of

$$f(x) = \frac{4x}{x^2 + 9}.$$

The domain of this function is $\mathbb{R}$, and it is continuous and differentiable everywhere. It is an odd function because

$$f(-x) = \frac{4(-x)}{(-x)^2 + 9} = -\frac{4x}{x^2 + 9} = -f(x),$$

so we only need to care about the region $x \geq 0$. As every odd continuous function $f(0) = 0$, and this is the only point where f croses the X axis. Besides $f(x) > 0$ for $x > 0$.

Its derivative is

$$f'(x) = \frac{4(x^2 + 9) - 4x \cdot 2x}{(x^2 + 9)^2} = \frac{4x^2 + 36 - 8x^2}{(x^2 + 9)^2} = \frac{4(9 - x^2)}{(x^2 + 9)^2}.$$

Thus, in $x \geq 0$ we have $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$. The function grows up to $x = 3$, where it has a local maximum, and then decreases beyond that point.

As for the second derivative,

$$\begin{aligned} f''(x) &= \frac{-8x(x^2 + 9)^2 - (36 - 4x^2)2(x^2 + 9)2x}{(x^2 + 9)^4} = \frac{-8x(x^2 + 9) - (36 - 4x^2)4x}{(x^2 + 9)^3} \\ &= \frac{8x^3 - 216x}{(x^2 + 9)^3} = \frac{8x(x^2 - 27)}{(x^2 + 9)^3}, \end{aligned}$$

so f is concave ($f'' < 0$) for $x < \sqrt{27} = 3\sqrt{3}$ and convex ($f'' > 0$) for $x > 3\sqrt{3}$. At $x = 3\sqrt{3}$ there is an inflection point.

Finally, there are no vertical asymptotes (f is defined in the whole $\mathbb{R}$), but since $\lim_{x \to \infty} f(x) = 0$, the X axis is a horizontal asymptote.

$f(x)$ is represented in Figure 4.5.

Exercises

Exercise 4.1 Let $f(x) = |x^3(x-4)| - 1$.

(a) Find where f is continuous and where it is differentiable.

(b) Determine its extrema.

(c) Prove that $f(x) = 0$ has a unique solution in $[0, 1]$.

Exercise 4.2 Solve these optimisation problems:

(a) A factory that produces tomato sauce wants to can it in cylindrical cans of a fixed volume V. Determine their radius r and height h so that their fabrication consumes the least possible material.

(b) A recipient with square bottom and no cap must be covered by a thin layer of lead. If the volume of the recipient must be 32 litres, which dimensions should it have so that it requires the least possible amount of lead?

(c) Find two numbers $x, y > 0$ such that $x + y = 20$ and x^2y^3 is maximum.

(d) Find the rectangle inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$ with its sides parallel to the axes of the ellipse, such that its area is maximum.

(e) With a tangent to the parabola $y = 6 - x^2$ and the positive axes one can make a triangle. Determine which of those triangles has the smallest area and compute it.

(f) We need to construct a box with no cap with the shape of a parallelepiped whose base is an equilateral triangle, and whose volume is 128 cm^3. If the material for the base costs 0.20 euros/cm^2 and that for the lateral surfaces costs 0.10 euros/cm^2, what are the dimensions of the cheapest such box?

(g) A right triangle ABC has vertex A at the origin, vertex B on the circumference $(x-1)^2 + y^2 = 1$ —side AB is the hypothenuse of the triangle— and side AC on the horizontal axis. Calculate the location of C that maximises the area of the triangle.

(h) Let $\mathrm{P} = (x_0, y_0)$ be a point of the first quadrant $(x_0, y_0 > 0)$. A straight line through P cuts the axes at $\mathrm{A} = (x_0 + \alpha, 0)$ and $\mathrm{B} = (0, y_0 + \beta)$. Calculate $\alpha > 0$ and $\beta > 0$ so as to minimise

   (i) the length of segment AB;

   (ii) the sum of the lengths of OA and OB;

   (iii) the area of the triangle OAB.

   HINT: Triangle similarity implies $\beta = x_0y_0/\alpha$.

Exercise 4.3 Prove the following inequalities:

(a) $(1 + x)^a \geq 1 + ax$ for all $a \geq 1$, $x > -1$ (Bernoulli's inequality);

(b) $e^x \geq 1 + x$ for all $x \in \mathbb{R}$;

(c) $\dfrac{x}{1+x} \leq \log(1 + x) \leq x$ for all $x > -1$.

HINT: In all cases try to minimise the appropriate function.

Exercise 4.4 Determine the number of solutions of the following equations in the specified domains:

(i) $x^7 + 4x = 3$ in $\mathbb{R}$;
(ii) $x^5 = 5x - 6$ in $\mathbb{R}$;
(iii) $x^4 - 4x^3 = 1$ in $\mathbb{R}$;
(iv) $\sin x = 2x - 1$ in $\mathbb{R}$;
(v) $x^x = 2$ in $[1, \infty)$;
(vi) $x^2 = \log \frac{1}{x}$ in $(1, \infty)$.

HINT: In order to cross the x-axis twice, a differentiable function f has to first increase and then decrease (or vice versa). This means that its derivative has to be zero at a point c between any two roots.

Exercise 4.5 Calculate the Taylor polynomial $P_{4,0}(x)$ for $f(x) = 1+x^3 \sin x$. Given the result, does f have a local maximum, minimum or inflection point at $x = 0$?

Exercise 4.6 Prove that if f and g are twice differentiable, convex functions, and f is increasing, then $h = f \circ g$ is convex.

Exercise 4.7 Discuss the convexity of the following functions:

(i) $f(x) = (x-2)x^{2/3}$;
(ii) $f(x) = |x|e^{|x|}$;
(iii) $f(x) = \log(x^2 - 6x + 8)$.

Exercise 4.8

(i) Sketch the graph of the function $f(x) = x + \log|x^2 - 1|$.
(ii) Based on the previous graph, plot function $g(x) = |x| + \log|x^2 - 1|$ and $h(x) = \left|x + \log|x^2 - 1|\right|$.

Exercise 4.9 Sketch a plot of the following functions:

(i) $f(x) = e^x \sin x$;
(ii) $f(x) = \sqrt{x^2 - 1} - 1$;
(iii) $f(x) = xe^{1/x}$;
(iv) $f(x) = x^2 e^x$;
(v) $f(x) = (x-2)x^{2/3}$;
(vi) $f(x) = (x^2 - 1)\log\left(\dfrac{1+x}{1-x}\right)$;
(vii) $f(x) = \dfrac{x}{\log x}$;
(viii) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$;
(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$;
(x) $f(x) = \log[(x-1)(x-2)]$;
(xi) $f(x) = \dfrac{e^x}{x(x-1)}$;
(xii) $f(x) = 2\sin x + \cos 2x$;
(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$;
(xiv) $f(x) = \sqrt{|x-4|}$;
(xv) $f(x) = \dfrac{1}{1+e^x}$;
(xvi) $f(x) = \dfrac{e^{2x}}{e^x - 1}$;
(xvii) $f(x) = e^{-x} \sin x$;
(xviii) $f(x) = x^2 \sin \dfrac{1}{x}$.

Exercise 4.10 Draw the graph of the following functions:

(i) $f(x) = \min\{\log|x^3 - 3|, \log|x + 3|\}$;
(ii) $f(x) = \dfrac{1}{|x| - 1} - \dfrac{1}{|x - 1|}$;
(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$, $(a > 0)$;
(iv) $f(x) = x\sqrt{x^2 - 1}$;

(v) $f(x) = \arctan \log |x^2 - 1|$;

(vi) $f(x) = 2 \arctan x + \arcsin \left(\dfrac{2x}{1 + x^2} \right)$.

Integration | 5

Integration is a device that was invented to calculate areas of figures limited by curved sides. The idea can be traced back at least to Archimedes. He is well known —among many other things— for calculating the area of a circle of unit diameter, A, in terms of its perimeter, π, obtaining the celebrated formula $A = \pi/4$. He did that by using two sequence of polygons, both circumscribed to and inscribed in the circumference, and then taking the limit of the number of sides going to infinity (see Figure 5.1).

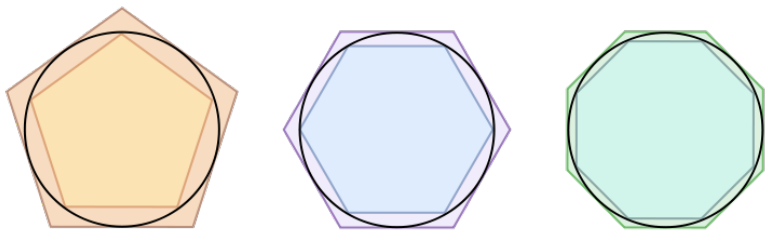


Figure 5.1: Archimedes's construction to obtain the relation between the area and the perimeter of a circle.

5.1 The Definite Integral

A similar idea was employed to obtain the area under more complicated curves. If we define a signed area as in Figure 5.2(a) (i.e., it adds if $f(x) > 0$ and substracts if $f(x) < 0$), the problem is how to calculate the total area enclosed by a curved within a given interval. Following Archimedes, one way to estimate that area is to approximate it as a sum of rectangles, as in Figure 5.2(b). In the limit when the width of these rectangles goes to zero we obtain the value of the seeked area.

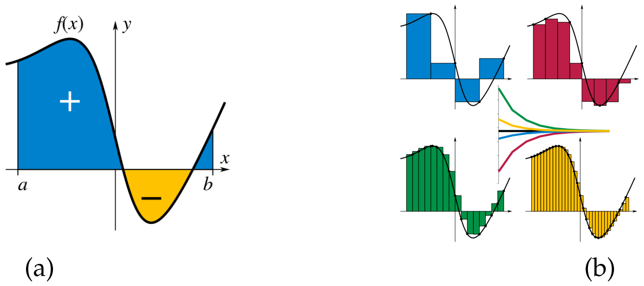


Figure 5.2: (a) Area "under" a curve: above the X axis area has a positive sign and below the X axis has a negative sign. (b) Approximations to that area as sums of thiner and thiner rectangles.

Example 5.1.1 As an example of this procedure, let us calculate, using this method, the area below the curve $f(x) = x^2$ within the interval $[0, a]$. To do that, we divide the interval in n rectangles of width a/n and heights $(ak/n)^2$, with $k = 1, 2, \ldots, n$. The areas of these rectangles will then be a^3k^2/n^3. This yields the following approximation to the area:

$$A_n = \frac{a^3}{n^3} + \frac{2^2a^3}{n^3} + \frac{3^2a^3}{n^3} + \cdots + \frac{n^2a^3}{n^3} = \frac{a^3}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right).$$

It is a know result that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

thus

$$A_n = a^3\frac{n(n+1)(2n+1)}{6n^3} = \frac{a^3(2n^3 + 3n^2 + n)}{6n^3}.$$

Therefore

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty}\left(\frac{a^3}{3} + \frac{a^3}{2n} + \frac{a^3}{6n^2}\right) = \frac{a^3}{3}$$

is the area we are seeking.

What we have found in the previous example is called the **Riemann integral**, because it was Riemann who developed this approach. Riemann's theory deals with more general partitions of a given interval, and not only on equidistant ones, but we will not see that here. So, as before, assume we divide the interval $[a, b]$ in n segments, each of width $(b - a)/n$. Then,

Definition 5.1.1 *(Definite integral) Let the points* $x_0 = a < x_1 < x_2 < x_3 < \cdots < x_n = b$ *divide the interval* $[a, b]$ *into n even subintervals, of width* $w = \frac{b-a}{n}$. *The definite integral of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\big(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big)\,w$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n-1} f(x_k)w$$

If the limit exists, then f *is said to be (Riemann) integrable on the interval* $[a, b]$.

It is customary to use Leibniz's notation $\int_a^b f(x)\,dx$ which reminds the definition of the integral as a sum (hence the sign $\int$) of the areas of rectagles of with dx and height $f(x)$, for all $a \le x \le b$.

We have seen that this limit exists for $f(x) = x^2$. What other functions are integrable?

Theorem 5.1.1 *If* f *is continuous in* $[a, b]$ *then it is integrable in* $[a, b]$.

The idea of the proof of this result is that continuous functions have the property that the difference between their maximum and minimum values in a closed interval is smaller the smaller the interval.

The class of functions that are Riemann integrable is quite a bit larger than the set of continuous functions; for instance, monotonous functions are integrable, but also functions that are both bounded (so that there exists an $M < \infty$ such that $|f(x)| < M$ for all x over which we wish to integrate) and piecewise continuous (continuous except for a finite number of discontinuities) are integrable. We will be concerned primarily with continuous functions in this text; knowing that continuous functions are Riemann integrable will therefore suffice for the most part.

Geometric interpretation of the integral

The geometric interpretation of the integral is straightforward: a definite integral can thus be interpreted as a difference of areas. If A_+ denotes the total area of the region above the x-axis and below the graph of f (where $f \geq 0$) and A_- denotes the total area of the region below the x-axis and above the graph of f (where $f \leq 0$), then

$$\int_a^b f(x)\,dx = A_+ - A_-.$$

Example 5.1.2 Find the value of $\int_0^{2\pi} \sin x\,dx$ by interpreting it as the signed area of an appropriately chosen region.

The function $f(x) = \sin x$ is symmetric about $x = \pi$. It follows from this symmetry that the area of the region below the graph of f and above the x-axis between 0 and π (denoted by A_+) is the same as the area of the region above the graph of f and below the x-axis between π and 2π (denoted by A_-). Therefore, $A_+ = A_-$ and

$$\int_0^{2\pi} \sin x\,dx = A_+ - A_- = 0.$$

5.2 Properties of the integral

Theorem 5.2.1 *Let f and g be two integrable functions in $[a,b]$. Then the following properties hold:*

(i) $\displaystyle\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ *for all* $\alpha, \beta \in \mathbb{R}$ ***linearity***

(ii) $\displaystyle\int_a^b f \leq \int_a^b g$ *whenever* $f \leq g$ *in* $[a,b]$ ***boundedness***

(iii) $|f|$ *is integrable in* $[a,b]$ *and* $\displaystyle\left|\int_a^b f\right| \leq \int_a^b |f|$ ***absolute integrability***

A consequence of (ii) is that if $f \geq 0$ then $\displaystyle\int_a^b f \geq 0$.

Another consequence is that if $M = \sup\limits_{x\in[a,b]} f(x)$ and $m = \inf\limits_{x\in[a,b]} f(x)$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a). \tag{5.1}$$

Theorem 5.2.2 (Interval additivity) *Given $a < b < c$, function f is integrable in $[a, c]$ if and only if it is integrable in $[a, b]$ and $[b, c]$. Besides*

$$\int_a^c f = \int_a^b f + \int_b^c f. \tag{5.2}$$

Notice that this formula implies

$$\int_a^b f = \int_a^c f - \int_b^c f,$$

so interval additivity will be preserved beyond the constraint $a < b < c$ if we *define*

$$\int_c^b f = -\int_b^c f. \tag{5.3}$$

Example 5.2.1 Show that

$$0 \leq \int_0^\pi \sin x \, dx \leq \pi$$

Note that $0 \leq \sin x \leq 1$ for $x \in [0, \pi]$. Using Property (6), we find that

$$\int_0^\pi \sin x \, dx \geq 0.$$

Using the properties of the integrals, we obtain

$$\int_0^\pi \sin x \, dx \leq (1)(\pi) = \pi.$$

5.3 The Fundamental Theorem of Calculus

The basic idea of the connection between integrals and derivatives —the essence of the fundamental theorem of calculus— is this. Let us denote $A(x)$ the (signed) area between the X axis and the function f within the interval $[a, x]$. Suppose that we increase the inverval by a very small amount h. In practical terms, we are enlarging the area by adding almost a rectangle of width h and height $\approx f(x)$. In other words,

$$A(x+h) \approx A(x) + f(x)h \qquad \Rightarrow \qquad f(x) \approx \frac{A(x+h) - A(x)}{h}.$$

If we now take the limit $h \longrightarrow 0$ we obtain the connection $A'(x) = f(x)$. This is the basic result that both Newton and Leibniz were aware of and which renders calculus such a powerful tool. The rigorous statement is as follows:

Theorem 5.3.1 (First fundamental theorem of calculus) *If f is continuous in $[a, b]$ then $F(x) = \int_a^x f(t)\, dt$ is differentiable in (a, b) and $F'(x) = f(x)$.*

The take-home message of this theorem is that integrals of functions are primitives of those functions. Here is the connection between differentiation and integration. From now on, calculating the area between the X axis and a given curve $f(x)$ is as simple as finding the right anti-derivative (also called primitive) of f. Actually, the problem is even easier: any primitive will do, because of this second version of the fundamental theorem of calculus:

Theorem 5.3.2 (Second fundamental theorem of calculus (Barrow's rule)) *If f is continuous in $[a, b]$ and G is any primitive of f in (a, b), then*

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Example 5.3.1 Evaluate

$$\int_{-1}^{2} (x^2 - 3x)\,dx.$$

Solution: Note that $f(x) = x^2 - 3x$ is continuous on $[-1, 2]$. We need to find an antiderivative of $f(x) = x^2 - 3x$; for instance, $F(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2$ is an antiderivative of $f(x)$ since $F'(x) = f(x)$. We then must evaluate $F(2) - F(-1)$:

$$F(2) = \frac{1}{3} \cdot 2^3 - \frac{3}{2} \cdot 2^2 = \frac{8}{3} - 6 = \frac{-10}{3},$$

$$F(-1) = \frac{1}{3} \cdot (-1)^3 - \frac{3}{2} \cdot (-1)^2 = \frac{-1}{3} - \frac{3}{2} = \frac{-11}{6}.$$

We find that

$$F(2) - F(-1) = \frac{-10}{3} - \left(\frac{-11}{6}\right) = \frac{-96}{6} = \frac{-32}{6} = \frac{-16}{3}.$$

Therefore,

$$\int_{-1}^{2} (x^2 - 3x)\,dx = F(2) - F(-1) = \frac{-16}{3}.$$

Table 5.1 shows some basic primitives that will be useful for our calculations.

Here are some important additional special cases:

$$\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + c, \qquad \int g'(x)[g(x)]^{\alpha}\,dx = \frac{g(x)^{\alpha+1}}{\alpha+1}, \quad \alpha \neq -1, \tag{5.4}$$

$$\int \frac{g'(x)}{1 + g(x)^2}\,dx = \arctan g(x) + c, \qquad \int \frac{g'(x)}{\sqrt{1 - g(x)^2}}\,dx = \arcsin g(x) + c. \tag{5.5}$$

$f(x)$	$F(x)$	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^α $(\alpha \neq -1)$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$\sin x$	$-\cos x$	$\dfrac{1}{1+x^2}$	$\arctan x$
x^{-1}	$\log\lvert x\rvert$	$\cos x$	$\sin x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
e^x	e^x	$\sinh x$	$\cosh x$	$\dfrac{1}{\cos^2 x}$	$\tan x$
a^x	$\dfrac{a^x}{\log a}$	$\cosh x$	$\sinh x$	$\dfrac{1}{\cosh^2 x}$	$\tanh x$

Table 5.1: Primitives $F(x)$ of some elementary functions $f(x)$ (up to the additive constant) as obtained by reversing Table 2.1. Here $\alpha \in \mathbb{R}$, $a > 0$.

Remark 5.3.1 Often primitives are referred to as "indefinite integrals" and denoted $\int f(x)\,dx$, whereas integrals of the form $\int_a^b f(x)\,dx$ are called "definite integrals".

Corollary 5.3.3 *If f is continuous in $[a,b]$ and g_1, g_2 are differentiable in (a,b) then*

$$H(x) = \int_{g_1(x)}^{g_2(x)} f(t)\,dt \tag{5.6}$$

is also differentiable in (a,b) and

$$H'(x) = f\big(g_2(x)\big)g_2'(x) - f\big(g_1(x)\big)g_1'(x). \tag{5.7}$$

Example 5.3.2 If

$$F(x) = \int_0^{x^3} \cos t\,dt,$$

then $F'(x) = 3x^2\cos(x^3)$.

Applying Barrow's rule we can obtain particular versions of the integration by parts and change of variable theorems:

Theorem 5.3.4 (Integration by parts) *If f and g are two differentiable functions in (a,b), then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx. \tag{5.8}$$

The symbol in the right-hand side is a short-hand for

$$f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a). \tag{5.9}$$

Theorem 5.3.5 (Change of variable) *If g is continuous in $[a,b]$ and differentiable in (a,b), and f is continuous in $g\big([a,b]\big)$, then*

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f\big(g(x)\big)g'(x)\,dx. \tag{5.10}$$

Example 5.3.3 Let us calculate the area of a circle of radius a. The equation of its circumference is $x^2 + y^2 = a^2$, from which we obtain $y = \pm\sqrt{a^2 - x^2}$. Clearly the area between the X axis and the function $f(x) = \sqrt{a^2 - x^2}$ within the interval $[-a, a]$ is half the area we want to calculate, therefore

$$A = 2\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx.$$

We can introduce the variable $t = x/a$, or $x = at$, so that $\dfrac{dx}{dt} = a$, and the limits $x = -a \;\longrightarrow\; t = -1$ and $x = a \;\longrightarrow\; t = 1$. Thus

$$A = 2\int_{-1}^{1} \sqrt{a^2 - a^2t^2}\, a\, dt = 2a^2\int_{-1}^{1} \sqrt{1 - t^2}\, dt.$$

Let us now introduce a second change of variable: $t = \sin\theta$. Then $\dfrac{dt}{d\theta} = \cos\theta$, and the limits $t = -1 \;\rightarrow\; \theta = -\pi/2$ and $t = 1 \;\rightarrow\; \theta = \pi/2$. The integral then becomes

$$\begin{aligned} A &= 2a^2\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = a^2\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta \\ &= a^2\Bigg(\pi + \underbrace{\frac{1}{2}\sin 2\theta\Big|_{-\pi/2}^{\pi/2}}_{=0}\Bigg) = \pi a^2. \end{aligned}$$

5.4 Applications of the integral

Cumulative change

Consider a population whose size at time t, $t \geq 0$, is $N(t)$ and that grows at a rate $r(t)$. Referring back to interpretations of the derivative in Section 2.2, we can say that:

$$\frac{dN}{dt} = r(t) \tag{5.11}$$

because $\frac{dN}{dt}$ is the rate of growth of the population represented using derivatives. Using the Fundamental Theorem of Calculus, we can see that, since $N(t)$ is an antiderivative of the function $r(t)$,

$$N(t) = \int_0^t r(s)\, ds + C \tag{5.12}$$

for some constant C. (We choose 0 as the lower limit of integration for convenience—the lower limit is arbitrary, and any value could be used.) If we know the function $r(t)$, then (5.12) can be used to calculate $N(t)$, but this only gives us $N(t)$ up to an unknown constant C. To calculate C, we need to know the value of $N(t)$ at a specific time; that is, we need an initial condition. The initial condition, say $N(0) = N_0$ for some known value of N_0, and the differential equation (5.11) together make an initial value problem

$$\frac{dN}{dt} = r(t) \quad \text{and} \quad N(0) = N_0$$

whose solution is:

$$N(t) = \int_0^t r(s)\,ds + N_0. \tag{5.13}$$

Example 5.4.1 A population grows at rate $r(t) = t^2/2$ and at time $t = 0$ contains 200 individuals. Find the size of the population (number of individuals) as a function of time.

Let $N(t)$ represent the size of the population at time t. Then we are given that

$$\frac{dN}{dt} = \frac{1}{2}t^2$$

and

$$N(0) = 200.$$

We can solve this initial value problem using (5.13):

$$\begin{aligned} N(t) &= \int_0^t \frac{1}{2}s^2\,ds + 200 \\ &= \frac{1}{6}t^3 + 200. \end{aligned}$$

Averages

To find the average value of a function between two points a and b, we can take samples of the function at equal distances. To formulate this notation mathematically, we evenly divide $[a, b]$ using $n + 1$ points: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with each pair of points separated by a distance $w = \frac{b-a}{n}$ (that is, $x_k - x_{k-1} = w$ for $k = 1, 2, \ldots, n$). We then measure the concentration at the right extreme of the interval, that is, at $x_1, x_2, \ldots, x_n$. The average value of f is then

$$\overline{f} = \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n))$$

This is similar to our equation for the integral, defined in Section 5.1 using sums. If we multiply and divide by $w = (b - a)/n$, we have

$$\overline{f} = \frac{w}{wn}(f(x_1)+f(x_2)+\cdots+f(x_n)) = \frac{1}{b-a}w(f(x_1)+f(x_2)+\cdots+f(x_n)),$$

since $wn = b - a$. When $n \to \infty$ the product on the right becomes an integral (see 5.1.1) and:

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

That is, the average concentration can be expressed as an integral over $c(x)$ between a and b, divided by the length of the interval $[a, b]$:

Definition 5.4.1 *(Average Value of a Function) Assume that $f(x)$ is a continuous function on $[a, b]$. The average value of f on the interval $[a, b]$ is*

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 5.4.2 Rainfall in Los Angeles varies seasonally, with most rain occurring at the beginning and end of the year. We might model this seasonal variation using the following formula for the monthly precipitation (in inches/month):

$$p(t) = 1.6 + 1.6\cos(2\pi t)$$

where t is the fraction of the year elapsed since January 1, so $t = 0$ is January 1, $t = 0.5$ is exactly halfway through the year (which turns out to be July 2), $t = 0.75$ is three- quarters of the way through the year (which turns out to be October 1). What is the average monthly rainfall in one year?

We use the definition of average usgin integrals:

$$\begin{aligned} P &= \frac{1}{1-0}\int_0^1 (1.6 + 1.6\cos(2\pi t))\,dt \\ &= 1.6\int_0^1 dt + 1.6\int_0^1 \cos(2\pi t)\,dt \\ &= 1.6 + 1.6\left[\frac{\sin(2\pi t)}{2\pi}\right]_0^1 \end{aligned}$$

Since $\sin(0) = \sin(2\pi) = 0$, we have

$$P = 1.6.$$

Exercises

Exercise 5.1

(a) Prove that if f is odd then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

(b) Prove that if f is even then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(c) Calculate the integral $\displaystyle\int_{6}^{10} \sin\left(\sin\left((x-8)^3\right)\right)dx$.

Exercise 5.2 Approximate the following integrals using n equal subintervals:

(a) $\int_{-1}^{1}(1-x^2)\,dx$ with $n=5$ subintervals.
(b) $\int_{-1}^{2} e^{-x}\,dx$ with $n=3$ subintervals.
(c) $\int_{0}^{\pi} \sin x\,dx$ with $n=4$ subintervals.

In all cases, compare the result with the exact one.

Exercise 5.3 Use an area formula from geometry to find the value of each integral by interpreting it as the (signed) area under the graph of an appropriately chosen function.

(a) $\int_{-3}^{3}|x|\,dx$
(b) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$
(c) $\int_{2}^{5}(\frac{x}{2}-4)\,dx$
(d) $\int_{-1}^{2}(2-|x|)\,dx$

Exercise 5.4 Calculate $F(x) = \displaystyle\int_{-1}^{x} f(t)\,dt$, with $-1 \le x \le 1$, for the following functions:

(i) $f(x) = |x-1/2|$;

(ii) $f(x) = \begin{cases} -1, & -1 \le x < 0, \\ 1, & 0 \le x \le 1; \end{cases}$

(iii) $f(x) = \begin{cases} x^2, & -1 \le x < 0, \\ x^2-1, & 0 \le x \le 1; \end{cases}$

(iv) $f(x) = \begin{cases} 1, & -1 \le x \le 0, \\ x+1, & 0 < x \le 1; \end{cases}$

(v) $f(x) = \begin{cases} 1+x, & -1 \le x \le -\frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1; \end{cases}$

Exercise 5.5 Calculate the derivative of the following functions:

(i) $F(x) = \displaystyle\int_{x^2}^{x^3} \frac{e^t}{t}\,dt$;

(ii) $F(x) = \displaystyle\int_{-x^3}^{x^3} \frac{dt}{1+\sin^2 t}$;

(iii) $F(x) = \displaystyle\int_{0}^{x} x^2 f(t)\,dt$, with f continuous in $\mathbb{R}$;

Exercise 5.6 Find the absolute maximum and minimum in the interval $[1,\infty)$ of the function

$$f(x) = \int_{0}^{x-1}\left(e^{-t^2} - e^{-2t}\right)dt.$$

HINT: $\lim\limits_{x\to\infty} \int_0^x e^{-t^2}\,dt = \sqrt{\pi}/2$.

Exercise 5.7 Prove that the equation

$$\int_0^x e^{t^2}\,dt = 1$$

has a unique solution in $\mathbb{R}$ and that it can be found in the interval $(0, 1)$.

Exercise 5.8 Let $f(x)$ be a continuous function such that $f(x) > 0$ for all $0 \leq x \leq 1$, and consider the function

$$F(x) = 2\int_0^x f(t)\,dt - \int_x^1 f(t)\,dt.$$

Determine how many solutions the equation $F(x) = 0$ has in $[0, 1]$.

Exercise 5.9 Let $f(x) = \displaystyle\int_{-1/x}^{x} \frac{dt}{a^2 + t^2}$. Determine, without computing the integral, for which values of a the function f is constant.

Exercise 5.10

(a) Use the change of variable $t = \sin^2\theta$ to calculate the integral

$$\int_0^1 \arcsin\sqrt{t}\,dt.$$

(b) Consider the function

$$f(x) = \int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \arccos\sqrt{t}\,dt.$$

Prove that $f(x) = c$, a constant, in the interval $[0, \pi/2]$.

(c) Determine the value of the constant c.

Exercise 5.11 Let $f : [-1, 1] \mapsto \mathbb{R}$ be any integrable function.

(a) Prove that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$$

HINT: Do the change of variables $y = \pi - x$.

(b) Calculate the integral

$$\int_0^\pi \frac{x\sin x}{1 + \cos^2 x}\,dx.$$

Exercise 5.12 Consider a population whose size at time t is $N(t)$ and whose growth obeys the initial-value problem

$$\frac{dN}{dt} = e^{-t}$$

with $N(0) = 100$.

(a) Find $N(t)$ by solving the initial-value problem.

(b) Compute the cumulative change in population size between $t = 0$ and $t = 5$.

Exercise 5.13 A particle moves along the x-axis with velocity

$$v(t) = -(t-2)^2 + 1$$

for $0 \leq t \leq 5$. Assume that the particle is at the origin at time 0.

(a) Use the graph of $v(t)$ to determine when the particle moves to the left and when it moves to the right.
(b) Find the location $s(t)$ of the particle at time t for $0 \leq t \leq 5$. Give a geometric interpretation of $s(t)$ in terms of the graph of $v(t)$.

Exercise 5.14 The average daily temperature (measured in Fahrenheit) in New York City can be approximated by the following function of the time of year t. (Here, t measures the fraction of the year that has elapsed since January 1.)

$$T(t) = 57.5 - 22.5\cos(2\pi t).$$

(a) What is the average daily temperature averaged over the course of one year?
(b) Explain how you could get your answer in part (b) without doing any integrations.
(c) What is the average summer temperature? You may assume that summer corresponds to the interval $0.47 \leq t \leq 0.73$. You will need to use a calculator to evaluate your answer.

Differential Equations | 6

We have already seen some examples of differential equations. In this chapter we will learn how to study them in more depth.

We will start with differential equations of the form:

$$\dot{x} = \frac{dx}{dt} = f(x)$$

Here $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x.[1] We'll call such equations one-dimensional or first-order systems.

1: Note that in this chapter we have switched our notation for the derivative with respect to t to $\dot{x}$, which is standard in books on dynamical systems.

We will not allow f to depend explicitly on time. Time-dependent or "nonautonomous" equations of the form $\dot{x} = f(x, t)$ are more complicated, because one needs two pieces of information, x and t, to predict the future state of the system.

6.1 Exponential Growth

We will start with the equation for exponential growth, that we derived in Section 2.2:

$$\dot{N} = \frac{dN}{dt} = rN, \tag{6.1}$$

where r is the per capita growth rate of the population.

This differential equation is called *separable* because we can put all the terms that are functions of N only on one side, and all the terms that are functions of t on the other. We can then integrate both sides:

$$\int \frac{dN/dt}{N}\, dt = \int r\, dt.$$

A primitive for the left-hand side is $\log N$, and so we have

$$\log N(t) = rt + C \implies N(t) = Ce^{rt}.$$

Now, in order to know the value of C we have to impose additional information.[2] In many cases, this additional information is the value of $N(t)$ at $t = 0$, which we call $N(0) = N_0$. A differential equation plus an initial condition is usually called an *initial value problem*. In order to do this, we take $t = 0$ in the solution and solve for C:

2: Because infinitely many functions have the same derivative rN.

$$N(0) = Ce^{r\cdot 0} \implies C = N_0.$$

Finally, the solution to the initial value problem is

$$N(t) = N_0 e^{rt} \tag{6.2}$$

If $r > 0$, the population grows exponentially without stop. However, if $r < 0$ the population decays exponentially until it gets to 0. We say that $N = 0$ is a **fixed point** of the system because $\dot{N}(0) = 0$. That is, the derivative does not change, and therefore the system remains in the same position forever. If $r > 0$, the population goes away from the fixed point, but if $r < 0$ it goes towards it.[3]

3: If $r = 0$ the population stays constant at N_0, because the derivative rN is always 0.

How can we understand the behavior of the system in general? Do we always need to find the solution of the differential equation?

6.2 A Geometric Way of Thinking

Pictures are often more helpful than formulas for analyzing nonlinear systems. Here we illustrate this point by a simple example. Along the way we will introduce one of the most basic techniques of dynamics: interpreting a differential equation as a vector field.[4]

4: We haven't introduced vectors yet, but perhaps you have seen them as "arrows" in high-school math. That idea will do for now.

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x. \tag{6.3}$$

To emphasize our point about formulas versus pictures, we have chosen one of the few nonlinear equations that can be solved in closed form. This differential equation is also separable, and so we can put all the terms that are functions of x only on one side, and all the terms that are functions of t on the other. We can then integrate both sides:

$$\int \frac{dx}{dt}\frac{1}{\sin x} dt = \int 1 dt,$$

which implies (you can look up the integral in a table)

$$t = -\log|\csc x + \cot x| + C$$

To evaluate the constant C, suppose that $x = x_0$ at $t = 0$. Then

$$C = \ln\left(\csc x_0 + \cot x_0\right).$$

Hence the solution is

$$t = \ln\left(\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right). \tag{6.4}$$

This result is exact, but a headache to interpret. For example, can you answer the following questions?

1. Suppose $x_0 = \frac{\pi}{4}$; describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?
2. For an arbitrary initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Think about these questions for a while, to see that formula (6.4) is not transparent. In contrast, a graphical analysis of (6.3) is clear and simple, as shown in Figure 6.1. We think of t as time, x as the position of an imaginary particle moving along the real line, and $\dot{x}$ as the velocity of that particle. Then the differential equation $\dot{x} = \sin x$ represents a vector field on the line: it dictates the velocity vector $\dot{x}$ at each x. To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the x-axis to indicate the corresponding velocity vector at each x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. This picture is often called *phase protrait*.

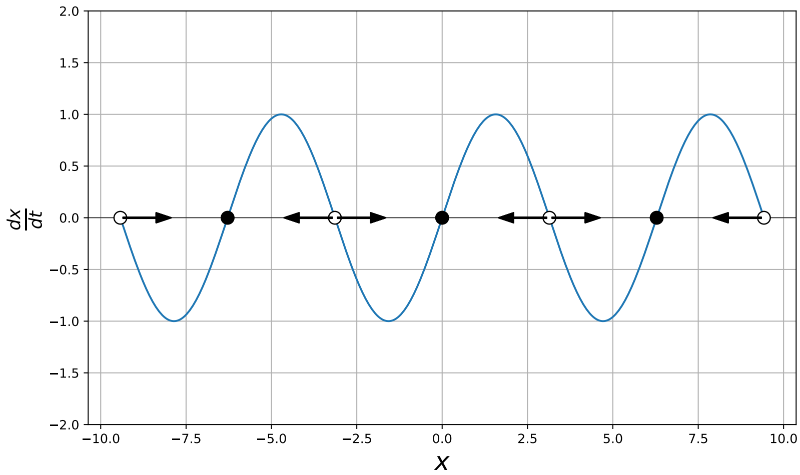


Figure 6.1: Phase protrait of the differential equation $\dot{x} = \sin x$, showing the fixed points (stable in black, unstable in white) and the direction of the flow.

Here's a more physical way to think about the vector field: imagine that fluid is flowing steadily along the x-axis with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. As shown in Figure 6.1, the flow is to the right when $x > 0$ and to the left when $x < 0$. At points where $\dot{x} = 0$, there is no flow; such points are therefore called fixed points. You can see that there are two kinds of fixed points in Figure 6.1: solid black dots represent stable fixed points (often called attractors or sinks, because the flow is toward them) and open circles represent unstable fixed points (also known as repellers or sources).

Armed with this picture, we can now easily understand the solutions to the differential equation $\dot{x} = \sin x$. We just start our imaginary particle at x_0 and watch how it is carried along by the flow.

This approach allows us to answer the questions above as follows:

1. Figure 6.1 shows that a particle starting at $x_0 = \frac{\pi}{4}$ moves to the right faster and faster until it crosses $x = \frac{\pi}{2}$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus, the qualitative form of the solution is as shown in Figure 6.2. Note that the curve is convex at first, and then concave; this corresponds to the initial acceleration for $x < \frac{\pi}{2}$ followed by the deceleration toward $x = \pi$.
2. The same reasoning applies to any initial condition x_0. Figure 6.1 shows that if $x > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point. Similarly, if $x < 0$ initially, the particle approaches the nearest stable fixed point to its left. If $x = 0$, then x remains constant.

A picture can't tell us certain quantitative things: for instance, we don't know the time at which the speed $\dot{x}$ is greatest. But in many cases qualitative information is what we care about, and then pictures are fine.

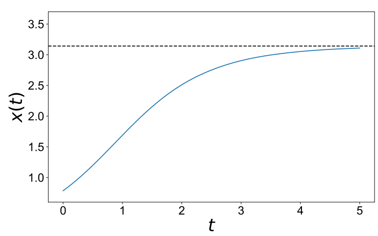


Figure 6.2: Solution of the differential equation $\dot{x} = \sin x$ with initial condition $x(0) = \pi/4$.

6.3 Fixed Points and Stability

The appearance of the phase portrait is controlled by the fixed points x^*, defined by $f(x^*) = 0$; they correspond to stagnation points of the flow. In Figure 6.1, the solid black dot is a stable fixed point (the local flow is toward it) and the open dot is an unstable fixed point (the flow is away from it).

In terms of the original differential equation, fixed points represent equilibrium solutions (sometimes called steady, constant, or rest solutions), since if $x = x^*$ initially, then $x(t) = x^*$ for all time. An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points.

Example 6.3.1 Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field. The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable.

Note that the definition of stable equilibrium is based on small disturbances; certain large disturbances may fail to decay. Here, all small disturbances to $x^* = -1$ will decay, but a large disturbance that sends x to the right of $x = 1$ will not decay—in fact, the phase point will be repelled out to $+\infty$. To emphasize this aspect of stability, we sometimes say that $x^* = -1$ is locally stable, but not globally stable.

Example 6.3.2 Sketch the phase portrait corresponding to $\dot{x} = x - \cos x$, and determine the stability of all the fixed points.

One approach would be to plot the function $f(x) = x - \cos x$ and then sketch the associated vector field. This method is valid, but it requires you to figure out what the graph of $x - \cos x$ looks like.

There's an easier solution, which exploits the fact that we know how to graph $g = x$ and $y = \cos x$ separately. We plot both graphs on the same axes and then observe that they intersect in exactly one point (do it!).

This intersection corresponds to a fixed point, since $x^* = \cos x^*$ and therefore $f(x^*) = 0$. Moreover, when the line lies above the cosine curve, we have $x > \cos x$ and so $\dot{x} > 0$: the flow is to the right. Similarly, the flow is to the left where the line is below the cosine curve. Hence x^* is the only fixed point, and it is unstable. Note that we can classify the stability of x^*, even though we don't have a formula for x^* itself!

6.4 Population Growth

The exponential growth seen in Equation (6.1) cannot go on forever. To model the effects of overcrowding and limited resources, population biologists and demographers often assume that the per capita growth rate $\frac{1}{N}\frac{dN}{dt}$ decreases when N becomes sufficiently large. For small N, the growth rate equals r, just as before. However, for populations larger than a certain carrying capacity K, the growth rate actually becomes negative; the death rate is higher than the birth rate.

A mathematically convenient way to incorporate these ideas is to assume that the per capita growth rate $\frac{1}{N}\frac{dN}{dt}$ decreases linearly with N.

This leads to the logistic equation first suggested to describe the growth of human populations by Verhulst in 1838:

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \tag{6.5}$$

Proposed Exercise 6.4.1 Solve the differential equation (6.5) analytically, taking the initial condition $N(0) = N_0$. The following fact will be useful:

$$\frac{1}{N(1 - N/K)} = \frac{1}{N} - \frac{1}{N - K}$$

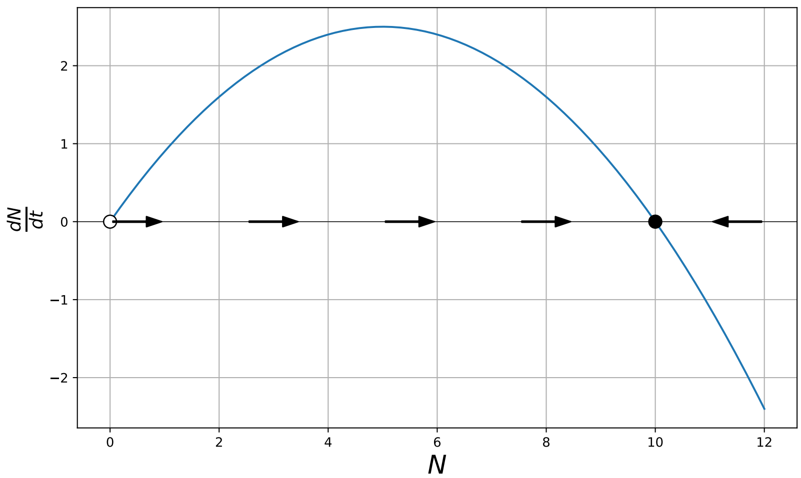


Figure 6.3: Phase protrait of the differential equation $\dot{N} = rN(1 - N/K)$, showing the fixed points (stable in black, unstable in white) and the direction of the flow.

Let us use our graphical approach again. We plot $\dot{N}$ versus N to see what the vector field looks like. Note that we plot only $N \geq 0$, since it makes no sense to think about a negative population (Figure 6.3). Fixed points occur at $N^* = 0$ and $N^* = K$, as found by setting $\dot{N} = 0$ and solving for N. By looking at the flow in Figure 6.3, we see that $N^* = 0$ is an unstable fixed point and $N^* = K$ is a stable fixed point. In biological terms, $N = 0$ is an unstable equilibrium: a small population will grow exponentially fast and run away from $N = 0$. On the other hand, if N is disturbed slightly from K, the disturbance will decay monotonically and $N(t) \longrightarrow K$ as $t \longrightarrow \infty$. In fact, all trajectories starting with $N(0) > 0$ will flow towards K.

The only exception is if $N_0 = 0$; then there's nobody around to start reproducing, and so $N = 0$ for all time. (The model does not allow for spontaneous generation!).

Figure 6.3 also allows us to deduce the qualitative shape of the solutions. For example, if $N_0 < \frac{K}{2}$, the phase point moves faster and faster until it crosses $N = \frac{K}{2}$, where the parabola in Figure 6.3 reaches its maximum. Then the phase point slows down and eventually creeps toward $N = K$. In biological terms, this means that the population initially grows in an accelerating fashion, and the graph of $N(t)$ is convex. But after $N = \frac{K}{2}$, the derivative $\dot{N}$ begins to decrease, and so $N(t)$ is concave as it asymptotes to the horizontal line $N = K$ (Figure 6.4).

Thus the graph of $N(t)$ is S-shaped or sigmoid for $N_0 < \frac{K}{2}$.

Something qualitatively different occurs if the initial condition N_0 lies between $\frac{K}{2}$ and K; now the solutions are decelerating from the start. Hence these solutions are concave for all t. If the population initially exceeds the carrying capacity ($N_0 > K$), then $N(t)$ decreases toward $N = K$ and is convex. Finally, if $N_0 = 0$ or $N_0 = K$, then the population stays constant.

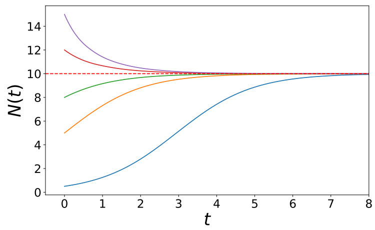


Figure 6.4: Trajectories of the logistic equation starting at different initial values.

6.5 Linear Stability Analysis

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by linearizing about a fixed point, as we now explain.

Let x^* be a fixed point, and let $q(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for q. Differentiation yields

$$\dot{q} = \dot{x} = f(x(t)) = f(x^* + q(t)).$$

Now using Taylor's expansion we obtain

$$f(x^* + q) = f(x^*) + f'(x^*)q + o(q),$$

where $o(q)$ denotes quadratically small terms in q. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\dot{q} = f'(x^*)q + o(q).$$

Now if $f'(x^*) \neq 0$, the $o(q)$ terms are negligible and we may write the approximation

$$\dot{q} \approx f'(x^*)q.$$

This is a linear equation in q, and is called the linearization about x^*. It shows that the perturbation $q(t)$ grows exponentially if $f'(x^*) > 0$ and decays if $f'(x^*) < 0$. If $f'(x^*) = 0$, the $o(q)$ terms are not negligible and a nonlinear analysis is needed to determine stability, as discussed in Example 6.5.3 below. The upshot is that the slope $f'(x^*)$ at the fixed point determines its stability. If you look back at the earlier examples, you'll see that the slope was always negative at a stable fixed point. The importance of the sign of $f'(x^*)$ was clear from our graphical approach; the new feature is that now we have a measure of how stable a fixed point is—that's determined by the magnitude of $f'(x^*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f'(x^*)|$ is a characteristic time scale; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

Example 6.5.1 Using linear stability analysis, determine the stability of the fixed points for

$$\dot{x} = \sin x.$$

The fixed points occur where $f(x) = \sin x = 0$. Thus $x^* = k\pi$, where k is an integer. Then

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & \text{k even} \\ -1, & \text{k odd} \end{cases}$$

Hence x^* is unstable if k is even and stable if k is odd. This agrees with the results shown in Figure 6.1.

Example 6.5.2 Classify the fixed points of the logistic equation, using linear stability analysis, and find the characteristic time scale in each case.

Here $f(N) = rN\left(1 - \frac{N}{K}\right)$, with fixed points $N^* = 0$ and $N^* = K$. Then

$$f'(N) = r\left(1 - \frac{2N}{K}\right),$$

and so $f'(0) = r$ and $f'(K) = -r$. Hence $N^* = 0$ is unstable and $N^* = K$ is stable, as found earlier by graphical arguments. In either case, the characteristic time scale is $1/|f'(N^*)| = 1/r$.

Example 6.5.3 If $f'(x^*) = 0$ nothing can be said about the stability of a fixed point in general. The stability is best determined on a case-by-case basis, using graphical methods. Consider the following examples:

(a) $\dot{x} = -x^3$
(b) $\dot{x} = x^3$
(c) $\dot{x} = x^2$
(d) $\dot{x} = 0$

Each of these systems has a fixed point $x^* = 0$ with $f'(x^*) = 0$. However the stability is different in each case. We can see graphically (Figure 6.5) that (a) is stable and (b) is unstable. Case (c) is a hybrid case we'll call half-stable, since the fixed point is attracting from the left and repelling from the right. We therefore indicate this type of fixed point by a half-filled circle. Case (d) is a whole line of fixed points; perturbations neither grow nor decay.

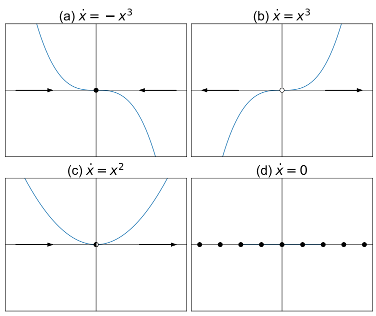


Figure 6.5: Stability of fixed points for differential equations where $f'(x^*) = 0$.

Exercises

Exercise 6.1 Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

(a) $\dot{x} = 4x^2 - 16$
(b) $\dot{x} = 1 - x^{14}$
(c) $\dot{x} = x - x^3$
(d) $\dot{x} = e^{-t}\sin x$
(e) $\dot{x} = 1 + \cos x$
(f) $\dot{x} = 1 - e^{\cos x}$
(g) $\dot{x} = e^x - \cos x$

Hint: In (g) sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.

Exercise 6.2 The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2,$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

(a) It can be shown that the analytical solution for $v(t)$, assuming that $v(0) = 0$, is

$$v(t) = \sqrt{\frac{mg}{k}}\,\frac{1 - e^{-2\sqrt{gk/m}t}}{1 + e^{-2\sqrt{gk/m}t}}.$$

Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

(b) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

Exercise 6.3 Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X,$$

in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production, a process called autocatalysis. This positive feedback process leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$. According to the law of mass action of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations by lowercase letters $x = [X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 ax - k_{-1}x^2,$$

where k_1 and k_{-1} are positive parameters called rate constants.

(a) Find all the fixed points of this equation and classify their stability.
(b) Sketch the graph of $x(t)$ for various initial values x_0.

Exercise 6.4 The growth of cancerous tumors can be modeled by the Gompertz law

$$\dot{N} = -aN\log(bN),$$

where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

(a) Interpret a and b biologically.
(b) Sketch the vector field and then graph $N(t)$ for various initial values.
(c) Using linear stability analysis, classify the fixed points of the model.

Exercise 6.5 For certain species of organisms, the per capita growth rate $\frac{\dot{N}}{N}$ is highest at intermediate N. This is called the Allee effect. For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large. One way to model this is to use

$$\frac{\dot{N}}{N} = r - a(N-b)^2, \qquad a, b, r > 0.$$

(a) Draw the per capita growth rate for this system, and dicuss how the behavior will change depending on whether the intercept (the value of $\dot{N}/N$ when $N = 0$) is positive or negative.
(b) Find all the fixed points of the system and classify their stability using linear stability analysis. Discuss how the value of the intercept affects the stability of the fixed points.
(c) Sketch the solutions $N(t)$ for different initial conditions.
(d) Compare the solutions $N(t)$ to those found for the logistic equation $\dot{N} = rN(1 - N/K)$. What are the qualitative differences, if any?

Exercise 6.6 Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

(a) $\dot{x} = x(1-x)$
(b) $\dot{x} = -x(1-2x)$
(c) $\dot{x} = \tan x$
(d) $\dot{x} = x^2(6-x^2)$
(e) $\dot{x} = 1 - e^{-x}$
(f) $\dot{x} = \log x$
(g) $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

Part II. Linear Algebra

Linear Functions of Several Variables 7

In this chapter we will take the first steps toward extending the methods of calculus from functions of a single variable to functions of multiple variables (also called multivariate functions) and to systems of functions. Our first step will be to consider an important class of multivariate functions: functions that are linear in several variables. We will learn how to solve equations involving these functions, and how to represent the solutions graphically.

7.1 Linear Transformations

So far, we have seen functions of one variable that have as an output another real number. However, in many real-world applications we will deal with functions of several variables.

Example 7.1.1 The function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (x^2+y^2, 2xz)$ is a function of several variables, that takes three variables as an input and outputs two variables. For instance, you can think of this function as giving the temperature and humidity of a spatial position in a room.

We call the elements of $\mathbb{R}^n$ **vectors**. Then, a function T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ a vector $T(\mathbf{x}) = (T_1(\mathbf{x}), \ldots, T_m(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the image of $\mathbf{x}$ (under the action of T). The set of all images $T(\mathbf{x})$ is called the range of T.

There are two elementary operations we can do with vectors:

1. **Sum:** Given two vectors in $\mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, their sum is defined as

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$

that is, we sum them component-wise.

2. **Multiplication by a scalar:** Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and a real number[1] $c \in \mathbb{R}$, the multiplication of $\mathbf{x}$ by c is given by

$$c\mathbf{x} = (cx_1, cx_2, \ldots, cx_n),$$

that is, we perform the multiplication component-wise.

1: Also called a *scalar* in this context.

In this chapter we will focus on a special case of multivariable functions, making use of these two elementary operations on vectors:

Definition 7.1.1 (Linear Transformations) *A function (also called transformation or mapping in this context) T is linear if:*

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v}$ *in the domain of* T*;*
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ *for all scalars* c *and all* $\mathbf{u}$ *in the domain of* T*.*

As a consequence of the above definition, we have that if T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d.[2]

2: Note that this means that the one-variable function $f(x) = 3x+5$ is not linear according to this definition, although it defines a line in the plane. In Linear Algebra, we only deal with lines that pass through zero.

Another way of writing this is that every linear transformation T can be written as follows:

$$T(\mathbf{x}) = T(x_1, \ldots, x_n) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}.$$

7.2 Solutions of Linear Equations

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we may want to know if the vector $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}^m$ is in the range of T, that is: is there any $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{b}$? And if there is, how many different $\mathbf{x}$ map to the same $\mathbf{b}$?[3]

3: You may be thinking that this is similar to the questions of injectivity and surjectivity that we saw for functions of one variable. You're right, they are the same.

Those questions amount to solving a system of linear equations. A linear equation in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the coefficients $a_1, \ldots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables. In the case of our linear transformation, for each component in the output vector $\mathbf{b}$ we have one linear equation.

Example 7.2.1 Given the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_1 + 4x_3)$, we may ask if the vector $\mathbf{b} = (1, 7)$ is in its image. This is the same as solving the system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 1 \\ x_1 + 4x_3 &= 7 \end{aligned}$$

A solution of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively. For instance, $(7, -13, 0)$ is a solution of the system because, when these values are substituted in for x_1, x_2, x_3, respectively, the equations are satisfied.

So $(1, 7)$ is in the range of T.

Note, however, that $(3, -4, 1)$ is also a solution: T is not injective. Are there more solutions to this system?

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Example 7.2.2 Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + 3x_2 &= 9 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2. A pair of numbers (x_1, x_2) satisfies both equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line.

Proposed Exercise 7.2.1 Draw the graphs that correspond to the following systems:

$$\text{(a)} \quad \begin{aligned} x_1 + 2x_2 &= 1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

$$\text{(b)} \quad \begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 2 \end{aligned}$$

and discuss how many solutions they have.

Therefore, a system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

7.3 Matrix Notation for Systems of Linear Equations

The essential information of a linear system can be recorded compactly in a rectangular array called a matrix. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 8x_3 &= 8 \\ 5x_3 &= 10 \end{aligned}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -8 \\ 0 & 0 & 5 \end{pmatrix}$$

is called the **coefficient matrix** of the system, and

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 0 & -8 & 8 \\ 0 & 0 & 5 & 10 \end{pmatrix}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$. An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.

The size of a matrix tells how many rows and columns it has. The augmented matrix above has 3 rows and 4 columns and is called a 3×4 (read "3 by 4") matrix. If m and n are positive integers, an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

7.4 Gaussian Elimination

Now we will describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve.

Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system:

Definition 7.4.1 (Elementary Row Operations) *The three elementary row operations that can be used to simplify a linear system are*

1. ***Replacement:*** *Replace one row by the sum of itself and a multiple of another row.*
2. ***Interchange:*** *Interchange two rows.*
3. ***Scaling:*** *Multiply all entries in a row by a nonzero constant.*

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other. It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a nonzero constant c, then multiplying the new row by $1/c$ produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that c times row 1 is added to row 2 to produce a new row 2. To "reverse" this operation, add c times row 1 to (new) row 2 and obtain the original row 2.

At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

Theorem 7.4.1 *If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.*

The algorithm is as follows:

Definition 7.4.2 (Gaussian Elimination) *Gaussian row reduction is a method used to solve systems of linear equations. The algorithm transforms the augmented matrix of the system into an upper trianguler matrix (with zeros below the diagonal) and, finally, into a unique reduced echelon form where the first nonzero entry of each row is equal to 1 and is the only nonzero entry of its column. From this reduced echelon form it is very easy to read the solutions of the system.*

1. ***Form the Augmented Matrix:***
   - *Write the augmented matrix of the system, which includes the coefficients of the variables and the constants on the right-hand side of the equations.*
2. ***Forward Elimination:***
   - *Begin with the leftmost nonzero column, which we call the **pivot column**.*
   - *Select a nonzero entry in the pivot column as the **pivot**. If necessary, interchange rows to place this pivot at the top of the current submatrix.*
   - *Use the pivot to create zeros below it in the pivot column. Do this by replacing each row below the pivot row with the sum of itself and a suitable multiple of the pivot row.*

- *Move to the next column to the right and repeat the process until all columns containing nonzero elements have been processed. The result is an upper triangular matrix (row echelon form).*

3. ***Backward Substitution (for reduced echelon form):***
   - *Start with the rightmost pivot and move to the left.*
   - *Normalize the pivot row by dividing it by the pivot element to make the pivot equal to 1.*
   - *Use the pivot to create zeros above it in the pivot column. Do this by replacing each row above the pivot row with the sum of itself and a suitable multiple of the pivot row.*
   - *Repeat this process for each pivot, moving from right to left, until the matrix is in reduced row echelon form.*

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system. Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent echelon form

$$\begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called basic variables. The other variable, x_3, is called a free variable.

Whenever a system is consistent, the solution set can be described explicitly by solving the reduced system of equations for the basic variables in terms of the free variables.

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

The statement "x_3 is free" means that you are free to choose any value for x_3. Once that is done, the formulas in (5) determine the values for x_1 and x_2. For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3.

Example 7.4.1 Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{pmatrix} 1 & 6 & 2 & 5 \\ 2 & 4 & 0 & 10 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The augmented matrix is not in upper triangular form, so We want to make zeros in each pivot column. The row reduction is completed next. The symbol ∼ before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$\begin{pmatrix} 1 & 6 & 2 & 5 \\ 2 & 4 & 0 & 10 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 2 & 5 \\ 0 & -8 & -4 & 0 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 2 & 5 \\ 0 & 1 & \frac{1}{2} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now the system can be solved easily for the basic variables, by starting from the bottom and back-substituting the values of the known variables in the upper equations. Note, however, that this system does not have any solutions. Why is that?

Theorem 7.4.2 (Existence and Uniqueness Theorem) *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & b \end{pmatrix}$$

with b nonzero.

If a linear system is consistent, then the solution set contains either

1. *a unique solution, when there are no free variables, or*
2. *infinitely many solutions, when there is at least one free variable.*

Exercises

Exercise 7.1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1 = (1,0)$ into $\mathbf{y}_1 = (2,5)$ and maps $\mathbf{e}_2 = (0,1)$ into $\mathbf{y}_2 = (-1,6)$. Find the images of $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Exercise 7.2 Solve the following systems of linear equations using Gaussian elimination:

1. $\begin{cases} 2x + y = 6 \\ x - 4y = -4 \end{cases}$
2. $\begin{cases} 5x + 2y = 8 \\ -x + 3y = 9 \end{cases}$
3. $\begin{cases} x - 2y + z = 3 \\ 2x - 3y + z = 8 \end{cases}$
4. $\begin{cases} 2x - y = 3 \\ x - y = 4 \\ x - 3y = 1 \end{cases}$
5. $\begin{cases} x + y = -1 \\ 2x - y = 7 \\ x - 2y = 8 \end{cases}$
6. $\begin{cases} 2x + z = 4y - 1 \\ x + 2y + 9 = 3z \\ 3x + 2z = 4 - 2y \end{cases}$
7. $\begin{cases} 5x - y + 2z = 6 \\ x + 2y - z = -1 \\ 3x + 2y - 2z = 1 \end{cases}$

Exercise 7.3 Find the general solutions of the systems whose augmented matrices are given in:

1. $\begin{pmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{pmatrix}$
4. $\begin{pmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{pmatrix}$
5. $\begin{pmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 1 \end{pmatrix}$
6. $\begin{pmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
7. $\begin{pmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
8. $\begin{pmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Exercise 7.4 Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

1. $\begin{pmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{pmatrix}$
2. $\begin{pmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{pmatrix}$

Exercise 7.5 Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

1. $\begin{cases} x_1 + hx_2 = 2 \\ 4x_1 + 8x_2 = k \end{cases}$
2. $\begin{cases} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k \end{cases}$

Exercise 7.6 Solve the following questions involving linear transformations:

1. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (3, 8)$.
2. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2) = (x_1 - 2x_2, x_1 + 3x_2, 3x_1 - 2x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (-1, 4, 9)$.

Matrix Algebra 8

8.1 Matrix Notation for Linear Transformations

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is an $m \times n$ matrix, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$, denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in $\mathbf{x}$.

This definition of matrix multiplication by a vector means that, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, for each $\mathbf{x}$ in $\mathbb{R}^n$, $T(\mathbf{x})$ can be computed as $A\mathbf{x}$, where A is an $m \times n$ matrix. For simplicity, we sometimes denote such a matrix transformation by $\mathbf{x} \mapsto A\mathbf{x}$. Observe that the domain of T is $\mathbb{R}^n$ when A has n columns and the codomain of T is $\mathbb{R}^m$ when each column of A has m entries. The range of T is the set of all linear combinations of the columns of A, because each image $T(\mathbf{x})$ is of the form $A\mathbf{x}$.

This means that every matrix defines a linear transformation, and that every linear transformation can be written in matrix form. So, in practice, the two mathematical objects are identical.

Example 8.1.1 Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 1 & 7 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ 2x_1 + 5x_2 \\ x_1 + 7x_2 \end{pmatrix}$$

(a) Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.
(b) Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.
(c) Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

(d) Determine if $\mathbf{c}$ is in the range of the transformation T.

Solution:

(a) Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1(2)+3(-1) \\ 2(2)+5(-1) \\ 1(2)+7(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -5 \end{pmatrix}$$

(b) Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

We row reduce the augmented matrix:

$$\begin{pmatrix} 1 & 3 & | & 1 \\ 2 & 5 & | & 2 \\ 1 & 7 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 1 \\ 0 & -1 & | & 0 \\ 0 & 4 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 1 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$$

Hence $x_1 = 1$, $x_2 = 0$, and $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The image of this $\mathbf{x}$ under T is the given vector $\mathbf{b}$.

(c) Since the solution of the previous equation is unique, there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$.

(d) The vector $\mathbf{c}$ is in the range of T if $\mathbf{c}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some $\mathbf{x}$. This is just another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{pmatrix} 1 & 3 & | & 3 \\ 2 & 5 & | & 4 \\ 1 & 7 & | & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 3 \\ 0 & -1 & | & -2 \\ 0 & 4 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & -6 \end{pmatrix}$$

The third equation, $0 = -6$, shows that the system is inconsistent. So $\mathbf{c}$ is not in the range of T.

Now, given a linear transformation T, what is the matrix A that defines it?

Proposed Exercise 8.1.1 If $T : \mathbb{R}_2 \longrightarrow \mathbb{R}_4$ is given by $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (5, 2, 0, 0)$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, what is the matrix A of T?

Remember that, for any $\mathbf{x} = (x_1, x_2)$, $T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2)$.

Theorem 8.1.1 (Matrix of a Linear Transformation) *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n.$$

The j-th column of A is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j-th column of the *identity matrix in $\mathbb{R}^n$:*

$$A = \begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{pmatrix}.$$

*The matrix A is called the **standard matrix** of the linear transformation T.*

Example 8.1.2 Let $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and T be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. T is called a *shear transformation* (Figure 8.1). It can be shown that if T acts on each point in the blue square (Figure 8.1, above), then the set of images forms the shaded red parallelogram (Figure 8.1, below). It can be shown that T maps line segments onto line segments, so we only need to check where the corners of the square are mapped to. For instance, the image of the point $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

$$T(\mathbf{u}) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

and the image of

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is

$$\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

This transformation deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and crystallography.

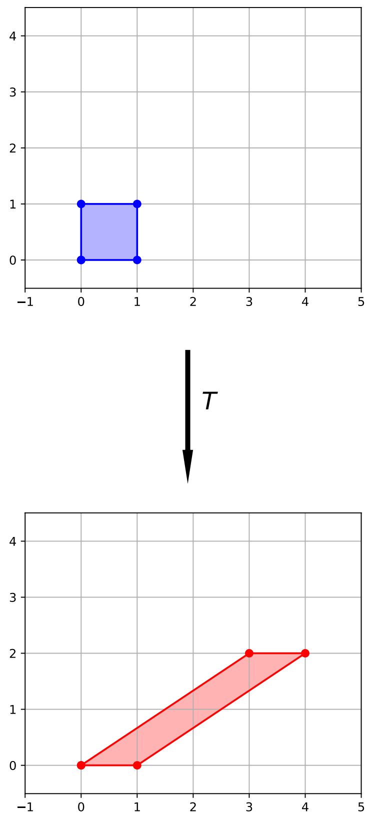


Figure 8.1: A shear transformation.

Example 8.1.3 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle φ, with counterclockwise rotation for a positive angle. We could show geometrically that such a transformation is linear. Find the standard matrix A of this transformation. Since

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ rotates into } \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ rotates into } \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}.$$

Therefore:

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

8.2 Operations with matrices

Matrix sum

The sum of two matrices A and B of the same dimensions $m \times n$ is obtained by adding their corresponding entries. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix},$$

then their sum $C = A + B$ is given by

$$C = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}.$$

Given two linear transformations $T_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $T_1(\mathbf{x}) = A\mathbf{x}$, and $T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, given by $T_2(\mathbf{x}) = B\mathbf{x}$, for matrices A and B with corresponding dimensions, their sum $S = T_1 + T_2$ is defined as $S :: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $S(\mathbf{x}) = C\mathbf{x}$, where the matrix $C = A + B$.

Matrix Multiplication

Matrix multiplication has an intuitive functional interpretation, that can be understood from the following example:

Example 8.2.1 Consider two linear transformations in $\mathbb{R}^2$. The transformation T_R reflects points across the line $x = y$, and its standard matrix R is:

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The transformation T_S scales each point away from the origin by a factor 3 in the horizontal direction and a factor 5 in the vertical direction. Its standard matrix S is:

$$S = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

If we apply T_S and then T_R we will obtain a new linear transformation, which is no other than the composition of these transformations, $T_R \circ T_S$. Since this is also a linear transformation, T_C, it will have its own standard matrix C.

In order to find C, we remember that its columns will be the transforms of $\mathbf{e}_1$ and $\mathbf{e}_2$. That is:

$$\mathbf{e}_1 \longrightarrow T_S(\mathbf{e}_1) = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{e}_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \rightarrow T_R(S\mathbf{e}_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

and

$$\mathbf{e}_2 \longrightarrow T_S(\mathbf{e}_2) = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{e}_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \longrightarrow T_R(S\mathbf{e}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Finally,

$$C = \begin{pmatrix} 0 & 5 \\ 3 & 0 \end{pmatrix}.$$

Note that the transformation of $\mathbf{e}_1$ by T_C, given by the product

$$\begin{pmatrix} 0 & 5 \\ 3 & 0 \end{pmatrix} \mathbf{e}_1,$$

is identical to the action of two matrices, first S and then R,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{e}_1 \right).$$

This is not only true for $\mathbf{e}_1$, but for every vector $\mathbf{x}$:

$$\begin{pmatrix} 0 & 5 \\ 3 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{x} \right).$$

Moreover, each column of C is the result of multiplying S by the corresponding column of R. This example suggests the following definition of matrix multiplication.

The product of an $m \times n$ matrix A and an $n \times p$ matrix B is an $m \times p$ matrix C whose entries are obtained by taking the dot product of the rows of A with the columns of B. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix},$$

then their product $C = AB$ is given by

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix},$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad \text{for all } i \text{ and } j.$$

Geometrically, the multiplication of two matrices A and B can be interpreted as the composition of two linear transformations T_A and T_B: the matrix $C = AB$ resulting from the multiplication of A and B corresponds to the matrix of a linear transformation T_C that is the composition of two transformations T_A and T_B, $T_C = T_A \circ T_B$.

Note that $C = AB$ implies that B is applied first, then A. In general, matrix multiplication is NOT commutative: the order in which we apply a linear transformation affects the result.

Proposed Exercise 8.2.1 What is the standard matrix of the transformation $S \circ T$ in example 8.2.1?

8.3 Determinants

Example 8.3.1 Given the linear transformation whose standard matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, we know that $\mathbf{e}_1$ will be expanded by a factor 3, while $\mathbf{e}_2$ will be expanded by a factor 2. Remembering example 8.1.2, we can ask what is the area of the parallelogram resulting from transforming the blue square in Figure 8.1. In this case, we have a 2×3 rectangle, with area equal to 6.

Note that knowing how the area of this particular square changes will tell us how the area of any region will change, since every square is affected similarly, and every area that is not square can be approximated by a sum of very small squares (and, eventually, by an integral!).

The quantity that tells us how the area of the square $[0, 1|] \times [0, 1]$ is scaled by a linear transformation whose standard matrix is A is called the **determinant** of A:

Definition 8.3.1 *(Determinant, 2×2 case) For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its determinant,* $\det A$ *is given by*

$$\det A = ad - cb.$$

The relationship between the formula of the determinant and it geomegtric interpretation is self-evident when $b = c = 0$, as the transformation of the $[0, 1] \times [0, 1]$ square is then a rectangle with sides of length a and d. When $b, c \neq 0$, however, this is not so easy to prove. See, however, exercise 8.8 for a geometric proof.

Example 8.3.2 Given the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & -3 \end{pmatrix}$, its determinant is -5.

Can you think of why the determinant is negative? Think about the orientation of the plane.

Proposed Exercise 8.3.1 Given the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, calculate its determinant. What does this mean? Think of a geometric interpretation.

For linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, the determinant gives us how much the volume of the cube $[0, 1] \times [0, 1] \times [0, 1]$ gets scaled. The rul for calculating the determinant is then:

Definition 8.3.2 (Determinant, 3×3 case) *For a* 3×3 *matrix*

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

the determinant $\det(A)$ *can be calculated using Sarrus's rule:*

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh.$$

Proposed Exercise 8.3.2 Show that the determinant of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 5 & 4 \\ 2 & 3 & 0 \end{pmatrix}$$

is -6.

The general rule to calculate the determinant of an $n \times n$ matrix is more complicated, and we will not cover it in this course.

8.4 Inverse of a Matrix

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, can we find another linear transformation T^{-1} such that $T \circ T^{-1} = T^{-1} \circ T = \mathrm{Id}$? Using our knowledge of matrix multiplication, if A is the standard matrix of T, finding the inverse of T amounts to finding a matrix C such that

$$AC = CA = I_n, \quad \text{where} \quad I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

The matrix I_n, with 1s on the diagonal and 0s elsewhere, is called the **identity matrix** of size n.

The matrix C is unique and we often refer to it as A^{-1}. Not all matrices (and therefore not all transformations) are invertible, and we will see in a moment when this happens.[1]

1: A matrix that is not invertible is sometimes called a singular matrix, and an invertible matrix is called a nonsingular matrix.

Here is a simple formula for the inverse of a 2×2 matrix, along with a test to tell if the inverse exists.

Definition 8.4.1 *(Inverse,* 2×2 *case)Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $ad - bc \neq 0$*, then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Example 8.4.1 Let

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$$

Then $\det A = -1 \neq 0$, and so A is invertible. Using the formula, we get

$$A^{-1} = -\begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}.$$

It is easy to check that the calculations are correct, since

$$AA^{-1} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

and

$$A^{-1}A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

What has the determinant got to do with the inverse? If T^{-1} is the inverse of T, then it has to "undo" the effect of T on $\mathbb{R}^n$, and so the determinant of its standard matrix should be the reciprocal of $\det A$. In other words,

$$\det A^{-1} = \frac{1}{\det A}.$$

Proposed Exercise 8.4.1 Check that the previous equation is true by doing the calculations with the matrices in example 8.4.1

So if $\det A = 0$ then the inverse is not defined. In this case, the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is collapsing one dimension of the plane onto one line, and thus many different inputs will end with the same output. In other words, this means that the linear transformation is not injective or one-to-one. So it is not possible to find an inverse! We can write this more rigorously as follows:

Theorem 8.4.1 *If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.*

Exercises

Exercise 8.1 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, with $T(e_1) = (1,3)$, $T(e_2) = (4,7)$, and $T(e_3) = (5,4)$, where e_1, e_2, and e_3 are the columns of the 3×3 identity matrix. Find the standard matrix of T.

Exercise 8.2 Given

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}, \quad E = \begin{pmatrix} -5 \\ 3 \end{pmatrix},$$

calculate:

1. $-2A$,
2. $B - 2A$,
3. AC,
4. CD,
5. $A + 2B$,
6. $3C - E$,
7. CB,
8. EB.

If an expression is undefined, explain why.

Exercise 8.3 Let $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -5 \\ 3 & k \end{pmatrix}$. What value(s) of k, if any, will make $AB = BA$?

Exercise 8.4 Let

$$A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix}.$$

Verify that $AB = AC$ and yet $B \neq C$.

Exercise 8.5 Find the inverses of the following matrices:

1. $\begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}$,
2. $\begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$,
3. $\begin{pmatrix} 8 & 5 \\ -7 & -5 \end{pmatrix}$,
4. $\begin{pmatrix} 3 & -4 \\ 7 & -8 \end{pmatrix}$.

Exercise 8.6 Solve the following systems:

1. $\begin{cases} 8x_1 + 6x_2 = 2, \\ 5x_1 + 4x_2 = 1. \end{cases}$
2. $\begin{cases} 8x_1 + 5x_2 = 9, \\ 7x_1 + 5x_2 = 11. \end{cases}$

In both cases, use the inverses found in Exercise 8.5.

Exercise 8.7 Given the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, explore the effect of an elementary row operation on $\det A$ by calculating the determinants of the following matrices. In each case, state the row operation and describe how it affects the determinant.

1. $\begin{pmatrix} c & d \\ a & b \end{pmatrix}$,

2. $\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$,
3. $\begin{pmatrix} a & b \\ kc & kd \end{pmatrix}$.

Exercise 8.8 Use Figure 8.2 to obtain the formula for the determinant of a 2×2 matrix using geometric arguments.

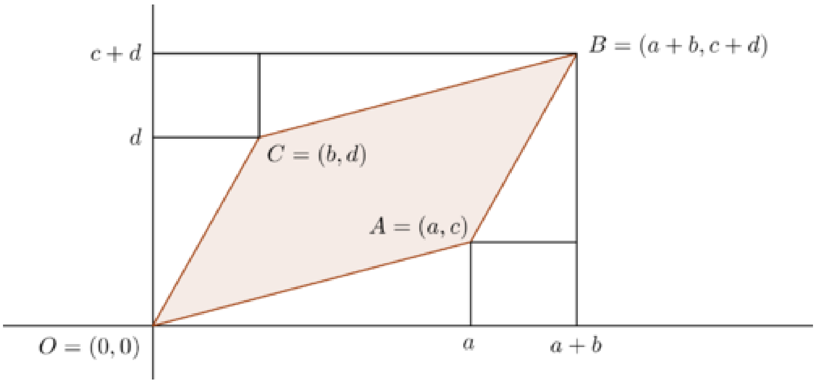


Figure 8.2: Exercise 8.8.

Exercise 8.9 Let S be the parallelogram determined by the vectors $\mathbf{b_1} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\mathbf{b_2} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and let $A = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Exercise 8.10 Given the two matrices $A = \begin{pmatrix} 6 & -1 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ -1 & -4 \end{pmatrix}$, obtain the matrix AB and calculate its determinant. Show that $\det(AB) = \det A \det B$. Give a geometric interpretation for this fact.

Exercise 8.11 Given the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, finding its inverse $A^{-1} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is the same as solving two systems of equations, namely:

$$\begin{cases} 2b_{11} + 5b_{21} = 1 \\ b_{11} + 3b_{21} = 0 \end{cases}$$

and

$$\begin{cases} 2b_{12} + 5b_{22} = 0 \\ b_{12} + 3b_{22} = 1. \end{cases}$$

Solve both systems at the same time by writing an augmented matrix with four columns: the two columns of A plus the two columns of the identity matrix. When the two leftmost columns are those of the identity, the two rightmost columns will be those of A^{-1}. Compare the result with the one you obtained using the formula.

9 Eigenvalues and Eigenvectors

In this chapter, we will only consider linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ and, in particular, only the case when $n = 2$. That is, we will be dealing with 2×2 matrices.[1] The goal of this chapter is to dissect the action of a linear transformation $\mathbf{x} \to A\mathbf{x}$ into elements that are easily visualized. The main motivation are dynamical systems, such as the one in the following example:

1: The theory we will develop is also valid for $n \times n$ matrices, although the computations can be much more involved.

Example 9.0.1 (A neuron-firing model) A very simple model of neuron activity is this: at a given (discrete) time step, a neuron can either be activated (firing) or in a resting state. If it is firing, it can either stop at the next step with probability 4/5, or remain firing with probability 1/5. If it is at rest, it can start firing with probability 1/3 or remain at rest with probability 2/3.

If we write the state of the neuron at time step n as

$$\mathbf{x}_n = (\text{probability of being firing, probability of being at rest}),$$

then we can calculate $\mathbf{x}_{n+1}$ as follows:

$$\mathbf{x}_{n+1} = \begin{pmatrix} 4/5 & 1/3 \\ 1/5 & 2/3 \end{pmatrix} \mathbf{x}_n.$$

That is, the probability that at time $n+1$ the neuron is firing is equal to the probability that it is firing at time n (this is $(\mathbf{x}_n)_1$ times the probability that it stays firing (4/5) plus the probability that it was at rest at time n times the probability that it starts firing (1/3). And similarly with the probability that it is at rest at time $n+1$.

The matrix $P = \begin{pmatrix} 4/5 & 1/3 \\ 1/5 & 2/3 \end{pmatrix}$ is called the **transition matrix** of this model.

Given this model, and supposing $\mathbf{x}_0 = (0, 1)$ (the neuron is at rest at time $n = 0$), what is the probability that it is firing at time $n = 1$? And at $n = 10$? And at $n = 100$?

We can calculate these numbers iterativelyy, as

$$\mathbf{x}_n = P\mathbf{x}_{n-1} = P^2\mathbf{x}_{n-1} = \cdots = P^n\mathbf{x}_0.$$

For instance, if $\mathbf{x}_0 = (1, 0)$, we have $\mathbf{x}_1 = (1/3, 2/3)$, and $\mathbf{x}_{10} \approx (0.624694, 0.375306)$. If we keep calculating with a computer, we observe that $\mathbf{x}_{100} \approx (0.625, 0.375)$, which is very similar to $\mathbf{x}_{10}$.

At the same time, if we calculate the product $P \cdot (0.625, 0.375)$ we obtain the same vector $(0.625, 0.375)$. What is happening? Can we understand this better?

9.1 Eigenvalues and Eigenvectors

In example 9.0.1 we have seen a matrix A and a vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$. This is a particular case of a general phenomenon:

Definition 9.1.1 (Eigenvalues and eigenvectors) *We say that* $\mathbf{v}$ *is an* ***eigenvector*** *or* A *if* $A\mathbf{v} = \lambda\mathbf{v}$ *for some number* λ*, which we call the* ***eigenvalue*** *associated with* $\mathbf{v}$*.*

Note that $\mathbf{v} = \mathbf{0}$ always satisfies the equation $A\mathbf{v} = \lambda\mathbf{v}$, for all λ, and so it is not very interesting!

This means that there are some directions that remain **invariant** under the action of the linear transformation associated with A: if $A\mathbf{v}$ is a multiple of $\mathbf{v}$, this means that every point in that line remains in that line, no matter what happens to the other points in $\mathbb{R}^n$.

It is very easy to show that a vector $\mathbf{v}$ is an eigenvector of a matrix A: just calculate $A\mathbf{v}$ and check that the result is proportional to $\mathbf{v}$.

Proposed Exercise 9.1.1 Let $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$. Are $\mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ eigenvectors of A?

It is also not very hard to check that a given number is an eigenvalue of a matrix (how would you do it?). However, it wouldn't be very smart to go checking every vector and number in order to see if they are eigenvectors and eigenvalues, so we will show a general method to do it.

Since eigenvectors and eigenvalues must satisfy the equation $A\mathbf{v} = \lambda v$, it must also be true that $A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$. In order to factor $\mathbf{v}$, we must multiply $\lambda\mathbf{v}$ by the identity matrix I, thus yielding $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

We saw in Section 8.4 that, if a matrix A is invertible, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. In particular, this means that the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution. But $\mathbf{x} = \mathbf{0}$ is always a solution of that system of equations,[2] so if there is a unique solution, it must be that one.

2: It is also called the *trivial* solution.

In the case that interests us now, the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ will have the unique solution $\mathbf{v} = \mathbf{0}$ if $A - \lambda I$ is invertible. But this is the case that we are not interested in: we want $\mathbf{v} \neq \mathbf{0}$.

For that to happen, the matrix $A - \lambda I$ must be singular or non-invertible. And the easiest way to check that is to show that its determinant is zero.

In other words: the eigenvalues of A are the numbers λ that satisfy

$$\det(A - \lambda I) = 0.$$

This equation is usually called the **characteristic equation** of A, and it is a polynomial equation in λ.

Once we have found the values of λ that satisfy the characteristic equation, we then substitute those values of λ into the system and solve it in search for the eigenvectors. Let's see some examples.

Example 9.1.1 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

We set up the characteristic equation $\det(A - \lambda I) = \mathbf{0}$:

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4.$$

Now we can easily solve this since it is a second-degree polynomial equation. The two solutions are $\lambda_1 = -1$ and $\lambda_2 = 4$.

Let's find the eigenvector associated to λ_1. We need to find the nonzero solutions of

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The general solution is $x_1 = -x_2$, with x_2 free. We can take as eigenvector $(1, -1)$, but note that there are infinite eigenvectors! All vectors proportional to $(1, -1)$ will also be eigenvectors of the same eigenvalue.

Now we find the eigenvectors associated with λ_2:

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_1 = \frac{2}{3} x_2.$$

So we could take $(2, 3)$ as eigenvector.

Good intuition comes from picturing what A does to the eigenvectors, and how that in turn deforms the whole plane.

When the eigenvalues are real, as in the previous example, all eigenvectors corresponding to a particular eigenvalue lie on the same straight line through the origin. For example, the line represented by the vector $(1, -1)$ is given by $l_1 = \{(x_1, x_2) : x_1 + x_2 = 0\}$, while the line represented by vector $(2, 3)$ is given by $l_2 = \{(x_1, x_2) : 3x1 - 2x_2 = 0\}$. The lines l_1 and l_2 are **invariant** under the map $\mathbf{x} \longrightarrow A\mathbf{x}$, in the sense that if we choose a point $(x1, x2)$ on a line that is represented by an eigenvector, then since $A\mathbf{x} = \lambda\mathbf{x}$ the result of the map is a point on that same line.[3]

3: Check this using the equation of the line.

Let's see another example:

Example 9.1.2 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

We set up the characteristic equation $\det(A - \lambda I) = \mathbf{0}$:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 1 = \lambda^2 - 2\lambda.$$

Now we can easily solve this since it is a second-degree polynomial equation. The two solutions are $\lambda_1 = 0$ and $\lambda_2 = 2$. Note that there is no problem with 0 being an eigenvalue: this just means that $\det A = 0$ but, as we already know, this means that the equations $A\mathbf{x} = \mathbf{0}$ has infinite nonzero solutions, aka eigenvectors.

Let's find the eigenvector associated to λ_1. We need to find the nonzero solutions of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The general solution is $x_1 = -x_2$, with x_2 free. We can take as eigenvector $(1, -1)$.

Now we find the eigenvectors associated with λ_2:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_1 = x_2.$$

So we could take $(1, 1)$ as eigenvector.

Here we can see that the line $x_1 + x_2 = 0$ is collapsed onto the origin, while the line $x_1 - x_2 = 0$ remains invariant.

Sometimes, the eigenvalues can be repeated:

Example 9.1.3 Find all the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

We set up the characteristic equation $\det(A - \lambda I) = \mathbf{0}$:

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda).$$

The solution is $\lambda = 2$, a double root of the polynomial. We say that the **algebraic multiplicity** of the eigenvalue is 2. Note that we can read the eigenvalues directly from the matrix: when the matrix is triangular, the determinant is equal to the product of the elements of the diagonal, and so the determinant is going to be zero whenever $\lambda - a_{ii} = 0$. In other words, for a triangular matrix the eigenvalues are the elements of the diagonal.

Are we going to find two eigenvectors associated with $\lambda = 2$? Let's find out! We need to find the nonzero solutions of:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The general solution is $x_2 = 0$, with x_1 free. We can take as eigenvector $(1, 0)$.

But there are no more solutions! There is only one direction of eigenvectors. We say that the **geometric multiplicity** of the eigenvalue is one. This is a very important fact for linear algebra problems but, sadly, we don't have time to go into it in this course.

9.2 Powers of a Matrix

For this discussion, we will restrict ourselves to the case in which A is a 2×2 matrix with real eigenvalues. We saw that in this case the eigenvectors define lines through the origin that are invariant under the map A. If the two invariant lines are not identical, we say that the two eigenvectors are **linearly independent**. This notion can be formulated as follows: two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ linearly independent if the equation $a\mathbf{u}_1 + b\mathbf{u}_2 = \mathbf{0}$ has no solution for any $a, b \in \mathbb{R}$.

In the case of eigenvectors, it is always the case that if $\lambda_1 \neq \lambda_2$ then the two eigenvectors are linearly independent. There are also cases in which $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent even though $\lambda_1 = \lambda_2$ (think of diagonal matrices with equal entries in the diagonal, for instance). We will, however, be concerned primarily with cases in which $\lambda_1 \neq \lambda_2$. Hence, the preceding criterion will suffice for our purposes. (The other cases are covered in courses on linear algebra.)

A consequence of linear independence is that we can write any vector uniquely as a linear combination of two eigenvectors. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent eigenvectors of a 2×2 matrix; then any vector $\mathbf{x} \in \mathbb{R}^2$ can be written as

$$\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2,$$

where $a_1, a_2 \in \mathbb{R}$ are uniquely determined. We will not prove this statement but will examine what we can do with it.

If we apply A to $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ we can use the linearity of A to show that

$$A\mathbf{x} = a_1 A\mathbf{u}_1 + a_2 A\mathbf{u}_2 = a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2,$$

the last step because $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A.

This representation of $\mathbf{x}$ is particularly useful if we apply A repeatedly. Applying A to $A\mathbf{x}$, we find that

$$A^2\mathbf{x} = A(a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2) = a_1\lambda_1 A\mathbf{u}_1 + a_2\lambda_2 A\mathbf{u}_2 = a_1\lambda_1^2\mathbf{u}_1 + a_2\lambda_2^2\mathbf{u}_2.$$

Continuing in this way yields

$$A^n\mathbf{x} = a_1\lambda_1^n\mathbf{u}_1 + a_2\lambda_2^n\mathbf{u}_2.$$

Thus, instead of multiplying A with itself n times (which is rather time consuming), we can use this equations, which just amounts to adding two vectors (a much faster task).

Example 9.2.1 (A neuron-firing model, continued) In Example 9.0.1 we saw that x_n could be calculated as $P^n\mathbf{x}_0$, where $P = \begin{pmatrix} 4/5 & 1/3 \\ 1/5 & 2/3 \end{pmatrix}$.

Let's calculate the eigenvalues and eigenvectors of this matrix:

$$\det(P - \lambda I) = \begin{vmatrix} 4/5 - \lambda & 1/3 \\ 1/5 & 2/3 - \lambda \end{vmatrix} = 0,$$

which yields the characteristic equation $15\lambda^2 - 22\lambda + 7 = 0$, with solutions $\lambda_1 = 1$ and $\lambda_2 = 7/15$.

The corresponding eigenvectors (do the calculations yourself) are: $\mathbf{u}_1 = (5, 3)$ and $\mathbf{u}_2 = (1, -1)$.

Then, $\mathbf{x}_0 = (0, 1) = a\mathbf{u}_1 + b\mathbf{u}_2$ is a system of linear equations with solution (do it!) $a = 1/8$, $b = -5/8$, and so:

$$\mathbf{x}_n = \begin{pmatrix} 5/8 \\ 3/8 \end{pmatrix} - \left(\frac{7}{15}\right)^n \begin{pmatrix} -5/8 \\ 5/8 \end{pmatrix}.$$

Note that $(7/15)^n$ becomes very small very rapidly (for instance, $(7/15)^{10} \approx 0.0005$) and therefore we can write

$$\mathbf{x}_n \approx \begin{pmatrix} 5/8 \\ 3/8 \end{pmatrix}$$

for n sufficiently large. This is what we had found originally!

Exercises

Exercise 9.1 Find all the eigenvalues and eigenvectors of the following matrices:

1. $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix}$
3. $\begin{pmatrix} -1 & 2 \\ 4 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

Exercise 9.2 For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there is a faster way to calculate eigenvalues, using the trace of A, $\operatorname{tr} A = a + d$ and its determinant $\det A = ad - bc$.

1. Show that the characteristic equation of A is $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$.
2. Since the characteristic equation will eventually be factores as $(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$, show that $\operatorname{tr} A = \lambda_1 + \lambda_2$ and $\det A = \lambda_1\lambda_2$.
3. Solve the system $\lambda_1 + \lambda_2 = \operatorname{tr} A, \lambda_1\lambda_2 = \det A$ to show that

$$\lambda_{1,2} = \frac{\operatorname{tr} A}{2} \pm \sqrt{\left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A}.$$

4. Use this formula to calculate the eigenvalues of the matrices in exercise 9.1.

Exercise 9.3 Let $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

1. Find all eigenvalues and eigenvectors of A.
2. Express $\mathbf{x} = (1, -3)$ as a linear combination of the eigenvectors of A.
3. Use the previous results to calculate $A^{20}\mathbf{x}$.

Exercise 9.4 Let $A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$. Find $A^{15}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ without using a calculator.

Exercise 9.5 Fibonacci proposed a model for the population growth of rabbits by means of an iterated discrete map: the number of young rabbits at time $n + 1$, Y_{n+1} is equal to the number of adult rabbits at time n, A_n (that is, every adult rabbit has one little rabbit every time step), while $A_{n+1} = A_n + Y_n$ as we assume that no rabbits die. The whole system can be written in matrix form as follows:

$$\begin{pmatrix} Y_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} Y_n \\ A_n \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

1. If $Y_0 = 1, A_0 = 0$, calculate $A_1, A_2, A_3, \ldots$. This list of numbers is called the **Fibonacci sequence**.
2. Check that you can also get the same Fibonacci sequence calculating $A_{n+2} = A_{n+1} + A_n$, starting from $A_0 = 0, A_1 = 1$. Try to reason why the two ways of expressing this problem are equivalent.
3. Find the eigenvalues and eigenvectors of the transition matrix T.
4. Since $(Y_n, A_n) = T^n(Y_0, A_0)$, use the eigenvalues and eigenvectors of T to calculate (Y_n, A_n) without calculating any power.

5. For large n, what is the fraction A_n/Y_n? [HINT: think of what happens to the smaller eigenvalue as n becomes very large.]

Exercise 9.6 A seabird colony consists of two classes of birds: immature birds that do not breed and adult birds that do breed. Assume that the number of immature birds is denoted by I_t and the number of mature birds by M_t. We model the changes in the sizes of two classes of birds from one year to the next using a Leslie matrix model:

$$\begin{pmatrix} I_{t+1} \\ M_{t+1} \end{pmatrix} = L \begin{pmatrix} I_t \\ M_t \end{pmatrix}, \qquad L = \begin{pmatrix} 0.5 & 2 \\ 0.3 & 0.9 \end{pmatrix}.$$

Show that the bird population is predicted to grow without bound and show that the ratio between M_t and Y_t becomes stable as $t \to \infty$.

Exercise 9.7 Denote the owl and wood rat populations at time k by $\mathbf{x}_k = (O_k, R_k)$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$\begin{pmatrix} O_{k+1} \\ R_{k+1} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{pmatrix} \begin{pmatrix} O_k \\ R_k \end{pmatrix} = \begin{pmatrix} 0.5O_k + 0.4R_k \\ -pO_k + 1.1R_k \end{pmatrix},$$

where p is a positive parameter to be specified. From the matrix we can see that with no wood rats for food, only half of the owls will survive each month, while with no owls as predators, the rat population will grow by 10% per month. If rats are plentiful, the $0.4R_k$ term will tend to make the owl population rise, while the negative term $-pO_k$ measures the deaths of rats due to predation by owls. (In fact, $1000p$ is the average number of rats eaten by one owl in one month.) Taking $p = 0.104$:

1. Find the eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of the transition matrix T.
2. If $\mathbf{x}_0 = a\mathbf{v}_1 + b\mathbf{v}_2$, write $\mathbf{x}_k = T^k\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
3. Determine the evolution of this system when $k \to \infty$.

Exercise 9.8 If an $n \times n$ matrix has n linearly independent eigenvectors, we say it is diagonalizable, because we can write

$$A = PDP^{-1},$$

where D is a diagonal matrix whose entries are the eigenvalues of A and P is the matrix whose columns are the eigenvectors of A.

For the following matrices, find the respective matrices D and P and check that $A = PDP^{-1}$:

1. $\begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}$
2. $\begin{pmatrix} a & 0 \\ 3(a-b) & b \end{pmatrix}$

Exercise 9.9 Diagonalization also gives an easy way to calculate the powers of a matrix:

1. Show that, if $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$.
2. Use this fact to calculate A^{10} for the matrices in Exercise 9.8.

Complex Numbers 10

As a way of motivation, we will see an example of a real 2×2 matrix that has no real eigenvalues:

Example 10.0.1 (Complex Eigenvalues) Find the eigenvalues of the matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the matrix of a rotation of angle $\pi/2$.

The characteristic equation is $\lambda^2 + 1 = 0$, which doesn't have a real solution: this is consistent with our expectation, as R does not leave any line invariant!

However, the fundamental theorem of algebra says that every polynomial equation of degree n has always n roots, counting multiplicities. But these roots will possibly be complex. In this example, there are no real eigenvalues, but the characteristic equations has two complex solutions, $+\sqrt{-1}$ and $-\sqrt{-1}$.

What does this mean for R? Before discussing this, we need to know more about complex numbers. .

10.1 Introduction to Complex Numbers

Even though it is sometimes said that complex numbers arise in order to find solutions to the equation $x^2 = -1$, this is not what really happened. Whenever a quadratic equation yielded complex solutions, authors concluded that there was no solution at all, because what they were really searching for was the intersection between the parabola and the x axis. In fact, complex numbers arose in the context of cubic equations. In 1545, Girolamo Cardano published a formula to solve the general cubic equation $x^3 = 3px + 2q$. The formula is

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}.$$

Notice that if $p^3 > q^2$ the solution involves dealing with square roots of negative numbers, but in this case the solutions could not be dismissed because they had an actual geometric meaning (the cubic always intersects the x axis). Almost thirty years after Cardano published his formula, Rafael Bombelli worked out the following example: putting $p = 5$, $q = 2$ in the above cubic expression leads to $x^3 = 15x + 4$, and the solution given by Cardano's formula is

$$x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

The actual solution is $x = 4$, and Bombelli realized that Cardano's formula would work if he could somehow prove that $\sqrt[3]{2 + 11i} = 2 + ni$ and $\sqrt[3]{2 - 11i} = 2 - ni$. In order for this to be true, he needed the sum of complex numbers to follow

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

Also, to find the n that made $\sqrt[3]{2 + 11i} = 2 + ni$, he needed to calculate $(2 + ni)^3$ and so he proposed the following multiplication rule

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc),$$

where he used $i^2 = -1$. Using these two rules, he was able to prove that $(2 \pm i)^3 = 2 \pm 11i$, thus solving the equation.

It was not until the end of the eighteenth century when complex numbers became prominent in mathematics. Wessel, Argand and Gauss independently gave a geometric interpretation to complex numbers, where $a + bi$ is the point in the xy-plane with Cartesian coordinates (a, b). The plane is now called the *complex plane*, denoted with the letter $\mathbb{C}$. In this geometric light, the sum and multiplication rules become:

> The sum $A + B$ of two complex numbers is given by the parallelogram rule of ordinary vector addition.
>
> The product AB is a vector whose length is the product of A and B, and whose angle is the sum of the angles of A and B,

where the angle of a vector is the one it makes with the x-axis. The correspondence between the algebraic and geometric rules for sum and multiplication is not hard to prove, and it will be useful to keep in mind in what follows.

10.2 Terminology and Notation

A complex number $z = x + iy$ is a point in the complex plane, and therefore has the same properties of a usual vector in $\mathbb{R}^2$. The length of z is usually called *modulus* and denoted $|z|$, and it is given by the usual formula $\sqrt{x^2 + y^2}$.

The angle z makes with the x axis is its *argument*, we denote it by $\arg z$.

The x coordinate of z is called its *real part* ($\operatorname{Re} z$) and the y coordinate is the *imaginary part* ($\operatorname{Im} z$).

Finally, the *complex conjugate* of z, denoted by $\overline{z}$, is the number given by $\overline{z} = x - iy$.

If we write $z = x + iy$ we say we are using the *binomial form*, but we could mark the same point in the plane by using its length $r = |z|$ and angle $\theta = \arg z$ as $x = r\angle\theta$. This is called the *polar form* of the number x, and it is especially useful when dealing with complex multiplication, as the geometric interpretation of complex multiplication yields

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = (r_1 r_2)\angle(\theta_1 + \theta_2).$$

There is not a unique polar form for each complex number. Because after turning 2π in a circle around the origin we go back to the point where we started, every complex number has infinitely many arguments, and therefore infinitely many polar forms. All of them, however, are summarized as

$$\arg z = \operatorname{Arg} z + 2\pi n, n \in \mathbb{Z},$$

where Arg $z \in [-\pi, \pi]$ is called the *principal argument* of z. This very simple fact about angles will become very important in the course of this subject.

10.3 Geometric interpretation of complex multiplication

It is not straightforward to understand the geometric meaning of complex multiplication from the binomial form that we have seen before. However, there is a trick. Note that the binomial form can also be written as $z = |z| \cos\theta + i|z| \sin\theta$. So if we take another complex number $w = |w| \cos\phi + i|w| \sin\phi$, the multiplication becomes

$$\begin{aligned} zw &= |z| \cdot |w| \left[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\right] \\ &= |z| \cdot |w| \left[\cos(\theta+\phi) + i\sin(\theta+\phi\right], \end{aligned}$$

and so the result is a complex number whose modulus is the product of the moduli of z and w and whose argument is the sum of the arguments of z and w. This connection will become much more intuitive in what follows:

10.4 Euler's Formula

Although we have just seen a way to write the polar form of the complex number $z = r\angle\theta$, we usually write it as $z = re^{i\theta}$. The equivalence between these two forms comes from the following formula, discovered by Leonhard Euler around 1740 (and thus called **Euler's formula**):

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The complex number $e^{i\theta}$ lies in the unit circle and has angle θ, as evidenced by the right-hand side of Euler's formula. As a result of this formula, complex multiplication now becomes

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)},$$

which is what we would have obtained from algebraically manipulating $e^{i\theta}$ using the rules for the real function e^x. We will see that, in fact, this is more than a coincidence.

Proof of Euler's formula using power series

In Chapter 3 we saw that the power series for the exponential function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

Euler wrote $x = i\theta$ in the above formula to obtain

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \dots$$

where we have used the fact that $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. Separating the real and imaginary parts of the right-hand side, we get

$$\operatorname{Re} e^{i\theta} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\operatorname{Im} e^{i\theta} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

and it is obvious from inspection of these power series that $\operatorname{Re} e^{i\theta} = \cos\theta$ and $\operatorname{Im} e^{i\theta} = \sin\theta$.

10.5 Complex Multiplication as Matrix Multiplication

When we multiply a given complex number $z = x + iy$ by another complex number $\lambda = a + bi$, the result is

$$\lambda z = (ax - by) + i(bx + ay).$$

But what if we could understand λ and z as a 2×2 matrix?

Consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ and their product:

$$z_1 z_2 = (a + bi)(c + di) = (ac - bd) + i(ad + bc) =: z.$$

Since we can interpret the numbers z_1 and z_2 as the vectors $(a, b) \in \mathbb{R}^2$ and $(c, d) \in \mathbb{R}^2$, respectively, what does it mean when we multiply z_1 by z_2? We haven't defined any multiplication of vectors by vectors where the result is a vector. But what if we could understand z_1 and z_2 as '2×2 matrices?

Let's define two matrices:

$$Z_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$Z_2 = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Note that these matrices store the same information as z_1 and z_2, respectively. Let's compute their matrix product:

$$Z_1 Z_2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{pmatrix} =: Z.$$

Comparing Z just above with z in Equation 3, we see that Z is indeed the matrix corresponding to the complex number $z = z_1 z_2$. Thus, we can represent any complex number z equivalently by the matrix:

$$Z = \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix},$$

and complex multiplication then simply becomes matrix multiplication. Further note that we can write:

$$Z = \mathrm{Re}(z) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{Im}(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

i.e., the imaginary unit i corresponds to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $i^2 = -1$ becomes:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Writing $z = re^{i\theta} = r(\cos\theta + i \sin\theta)$, we find

$$Z = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

corresponding to a stretch factor r multiplied by a 2D rotation matrix. In particular, multiplication by i corresponds to the rotation with angle $\theta = \pi/2$ and $r = 1$, which is what we had seen in Example 10.0.1.

Exercises

Exercise 10.1 Find real numbers x and y such that

$$\frac{43+iy}{x-i5} = 4+i3$$

Exercise 10.2 Compute

1. $(2-i)^3$,
2. i^{13},
3. $\frac{1}{i}$,
4. $\frac{1}{1+2i}$,
5. $\frac{1+i}{1-i}$,
6. $i+i^2+i^3+i^4$.

Exercise 10.3 Compute the complex conjugate of $z = \left(\frac{a+bi}{a-bi}\right)^2 + \left(\frac{a-bi}{a+bi}\right)^2$.

Exercise 10.4 Let $z \in \mathbb{C}$, find the real and the imaginary part of

1. $z+3i$,
2. iz,
3. $(1+z)(\bar{z}+1)$

in terms of the real and imaginary parts of z.

Exercise 10.5 Compute the modulus of the following complex numbers:

1. $-i$,
2. $1+i$
3. $1-i$
4. $(1+i)^2$
5. $\frac{1}{1+i}$,
6. $\frac{1}{(1-i)^2}$,
7. $1-i\sqrt{3}$.

Exercise 10.6 Find the principal argument of the following complex numbers, and express it in radians:

1. $1+i$
2. $(1+i)^{-1}$
3. $(1+i)^2$
4. $(1+i)^3$

[Remember, the principal argument of a complex number is the only argument θ that satisfies $-\pi \leq \theta \leq \pi$.]

Exercise 10.7 Express the following numbers in binomial form (i.e. as $a+bi$, where a and b are the real and the imaginary part, respectively):

1. $e^{\frac{\pi}{4}i} - e^{-\frac{\pi}{4}i}$,
2. $\frac{1-e^{\frac{\pi}{2}i}}{1+e^{\frac{\pi}{2}i}}$,
3. $e^{\pi i}(1-e^{-\frac{\pi}{3}i})$.

Exercise 10.8 Express the following numbers in polar form (i.e. as $re^{i\theta}$, where r is the modulus and θ is the principal argument):

1. $-i$,
2. $1+i$
3. $1-i$
4. $(1+i)^2$
5. $\frac{1}{1+i}$,
6. $\frac{1}{(1-i)^2}$,
7. $1-i\sqrt{3}$.

Exercise 10.9 We can use Euler's Formula to derive many relevant trigonometric identities.

1. Find an expression for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

2. Express $2^4 \cos^4 \theta$ in terms of cosines of multiples of theta. [HINT: $2\cos\theta = e^{i\theta} + e^{-i\theta}$.]

Exercise 10.10 On December 21, 1807, an engineer named Joseph Fourier announced to the prestigious French Academy of Sciences that an arbitrary function $f(x)$ could be expanded in an infinite series of sines and cosines. Specifically, let $f(x)$ be defined on the interval $-L \le x \le L$, and compute the numbers

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 0, 1, 2, \ldots \tag{1}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, \ldots \tag{2}$$

Then, the infinite series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \tag{3}$$

converges to $f(x)$.

Use Euler's formula to express the coefficients of the Fourier series as complex exponentials, so that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi x}{L}}.$$

What is the relationship between a_n, b_n and c_n?

Exercise 10.11 Find the eigenvalues of the following matrices and discuss the geometric action of the corresponding linear transformation.

1. $\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$
2. $\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 3 & 3 \\ -2 & 2 \end{pmatrix}$
4. $\frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Exercise 10.12 The equation $z = (re^{i\theta})^{1/n}$ has n complex roots, given by the following formulas:

$$z_k = r_k e^{i\theta_k}, \qquad r_k = r^{1/n}, \qquad \theta_k = \frac{\theta + 2\pi k}{n}, \qquad k = 0, 1, \ldots, n-1.$$

Find the following roots:

1. $(-i)^{1/2}$,
2. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{1/2}$,
3. $(-1)^{1/4}$,
4. $1^{1/6}$.

Part III. Systems of Differential Equations

Systems of Linear Differential Equations 11

In Chapter 6 we saw that differential equations in one-dimensional phase spaces were extremely confined— all trajectories are forced to move monotonically or remain constant. In higher-dimensional phase spaces, trajectories have much more room to maneuver, and so a wider range of dynamical behavior becomes possible. Rather than attack all this complexity at once, we begin with the simplest class of higher-dimensional systems, namely linear systems in two dimensions. These systems are interesting in their own right, and, as we'll see later, they also play an important role in the classification of fixed points of nonlinear systems. We begin with some definitions and examples.

11.1 Definitions and Examples

A two-dimensional linear system is a system of the form

$$\begin{cases} \dot{x} = ax + by, \\ \dot{y} = cx + dy, \end{cases}$$

where a, b, c, d are parameters. If we use boldface to denote vectors, this system can be written more compactly in matrix form as

$$\dot{\mathbf{x}} = A\mathbf{x},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such a system is linear in the sense that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then so is any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Notice that $\mathbf{x} = 0$ when $\mathbf{x} = 0$, so $\mathbf{x}^* = 0$ is always a fixed point for any choice of A.

The solutions of $\mathbf{x} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, in this context called the phase plane. Our first example presents the phase plane analysis of a familiar system.

Example 11.1.1 As discussed in elementary physics courses, the vibrations of a mass hanging from a linear spring are governed by the linear differential equation

$$m\ddot{x} + kx = 0, \tag{11.1}$$

where m is the mass, k is the spring constant, and x is the displacement of the mass from equilibrium. Let's give a phase plane analysis of this simple harmonic oscillator.

This system can actually be solved analytically. But that's precisely what makes linear equations so special! For the nonlinear equations of ultimate interest to us, it's usually impossible to find an analytical solution. We want to develop methods for deducing the behavior of equations like (11.1) without actually solving them.

The motion in the phase plane is determined by a vector field that comes from the differential equation (11.1). To find this vector field, we note that the state of the system is characterized by its current position x and velocity v; if we know the values of both x and v, then (11.1) uniquely determines the future states of the system. Therefore, we rewrite the system in terms of x and v, as follows:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\frac{k}{m}x. \end{cases}$$

The first equation is just the definition of velocity, and the second is the differential equation (11.1) rewritten in terms of v. To simplify the notation, let $\omega^2 = \frac{k}{m}$. Then the system becomes

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\omega^2 x. \end{cases}$$

This system assigns a vector $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$ at each point (x, v), and therefore represents a vector field on the phase plane.

For example, let's see what the vector field looks like when we're on the x-axis. Then $v = 0$ and so $(\dot{x}, \dot{v}) = (0, -\omega^2 x)$. Hence the vectors point vertically downward for positive x and vertically upward for negative x (Figure 11.1). As x gets larger in magnitude, the vectors $(0, -\omega^2 x)$ get longer. Similarly, on the v-axis, the vector field is $(\dot{x}, \dot{v}) = (v, 0)$, which points to the right when $v > 0$ and to the left when $v < 0$. As we move around in phase space, the vectors change direction as shown in Figure 11.1.

Just as in Chapter 6, it is helpful to visualize the vector field in terms of the motion of an imaginary fluid. In the present case, we imagine that a fluid is flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$. Then, to find the trajectory starting at (x_0, v_0), we place an imaginary particle or phase point at (x_0, v_0) and watch how it is carried around by the flow.

The flow in Figure 11.1 swirls about the origin. The origin is special, like the eye of a hurricane: a phase point placed there would remain motionless, because $(\dot{x}, \dot{v}) = (0, 0)$ when $(x, v) = (0, 0)$; hence the origin is a fixed point. But a phase point starting anywhere else would circulate around the origin and eventually return to its starting point. Such trajectories form closed orbits, as shown in Figure 11.1. Figure 11.1 is called the phase portrait of the system—it shows the overall picture of trajectories in phase space.

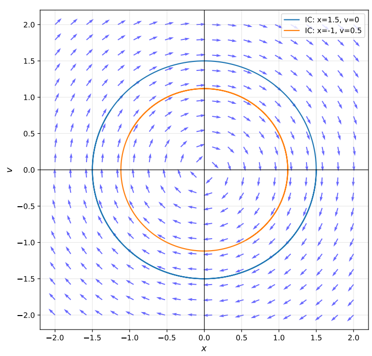


Figure 11.1: Vector field for the simple harmonic oscillator, with two trajectories plotted on top of it. The initial conditions are $x_0 = 1.5, v_0 = 0$ (blue line) and $x_0 = 1, v_0 = 0.5$ (orange line).

11.2 Solving Linear Systems

The trajectories plotted in Figure 11.1 can actually be obtained analytically. Given the system $\dot{\mathbf{x}} = A\mathbf{x}$, its solutions will be vector-valued functions:

that is, if $\mathbf{x}(t) = (x(t), y(t))$ we will have two functions. As in Chapter 6, we will see that every system of differential equations admits infinite solutions, that we will need to determine using initial conditions.[1] Now, we are going to try something crazy. Remember that the one-variable differential equation $\dot{x} = ax$ had as solution $x(t) = e^{at}$? Since the two-dimensional equation looks similar, maybe we can try a solution of the type $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where λ is a constant and $\mathbf{v}$ is a vector. If we plug it into the system,

$$\dot{\mathbf{x}}(t) = \lambda e^{\lambda t}\mathbf{v} = A\mathbf{x} = Ae^{\lambda t}\mathbf{v} \implies \lambda\mathbf{v} = A\mathbf{v}.$$

1: Note that now we need two initial conditions in order to completely determine the system.

So this works if λ is an eigenvalue of A and $\mathbf{v}$ its corresponding eigenvector. And since we know that A will (almost) always have two different eigenvalues,[2] and that if we have two different solutions their linear combination is also a solution[3] , we can write the general solution of the system as follows:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

2: The case where A has two equal eigenvalues is a bit different and we will not see it in this course.

3: This is sometimes called the *supeposition principle*

(Note that this is very similar to the discrete systems we saw in Section 9.2.

Example 11.2.1 Solve the following system of differential equations:

$$\begin{cases} \dot{x} = 2x - 2y \\ \dot{y} = 2x - 3y \end{cases}$$

The eigenvalues of the matrix $\begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}$ are $\lambda_1 = 1$, $\mathbf{v}_1 = (2, 1)$ and $\lambda_2 = -2$, $\mathbf{v}_2 = (1, 2)$. So the general equation for the system will be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The constants c_1 and c_2 will have to be determined by the initial conditions $x(0)$ and $y(0)$. For instance, if $x_0 = -1, y_0 = 4$, we make $t = 0$ in the solution above and solve the system

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which has as solution $c_1 = -2, c_2 = 3$. Try to get an intuitive understanding of what is happening with the trajectories by plotting the invariant lines and sketching the vector field (Figure 11.2).

11.3 Equilibria and Stability

We have seen that the zero vector is always a fixed point of the system $\dot{\mathbf{x}} = A\mathbf{x}$. If $\det A = 0$ there will be other fixed points:

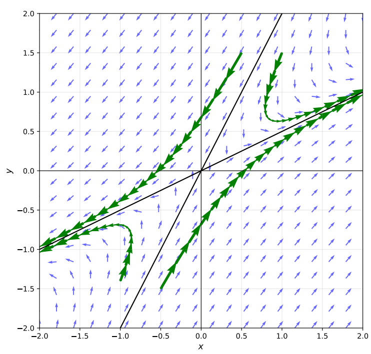


Figure 11.2: Vector field for the system in Example 11.2.1, with the invariant lines defined by the eigenvectors of the matrix A in black and some trajectories in green. Note that the trajectories approach the origin from one direction, but they go away from it from the other.

Example 11.3.1 Take the system

$$\begin{cases} \dot{x} = 3x + y \\ \dot{y} = 6x + 2y \end{cases}$$

It is obvious that $(0,0)$ is a fixed point, because $\dot{x}(0,0) = 0$ and $\dot{y}(0,0) = 0$. But we can also see that any point that satisfies $3x = -y$ will also be a fixed point. In this case, we say there exists a *degenerate* line of fixed points.

Let's assume $\det A \neq 0$. Then $(0,0)$ is the only solution of the system $A\mathbf{x} = \mathbf{0}$, that is, the only fixed point of the system. From the general solution for the linear system that we wrote above, we can see that the behavior of the trajectories as $t \to \infty$ will depend on the values of the eigenvalues of A.

First, it's useful to introduce some language that allows us to discuss the stability of different types of fixed points. This language will be especially useful when we analyze fixed points of nonlinear systems in Chapter 12. We say that $x^* = 0$ is an **attracting fixed point** if all trajectories that start near x^* approach it as $t \to \infty$. That is, $x(t) \to x^*$ as $t \to \infty$. In fact, x^* attracts all trajectories in the phase plane, so it could be called *globally attracting*. When trajectories go far from the fixed point when $t \to \infty$ we say that the fixed point is **unstable**. Finaly, if the trajectories don't go either toward nor away from the fixed point, we say it is **neutrally stable** (Figure 11.1).

11.4 Classification of Fixed Points

We can show the type and stability of all the different fixed points, depending on the eigenvalues:

Different real eigenvalues

If both eigenvectors are real but different, we have three cases:

1. If $\lambda_1, \lambda_2 < 0$ the origin is stable. The trajectories will approach it getting closer to the invariant lines defined by the eigenvectors. In this case we say the origin is a **stable node**.
2. If $\lambda_1, \lambda_2 > 0$ the origin is unstable. The trajectories will go away from it getting closer to the invariant lines defined by the eigenvectors. Here we say the origin is an **unstable node**.
3. If $\lambda_1 < 0$ and $\lambda_2 > 0$, we have a **saddle point**, such as in Example 11.2.1. Here the origin is still unstable, but one of the directions is attracting and so it merits a special mention.

Complex eigenvalues

If the eigenvalues are complex $\lambda_{1,2} = a \pm bi$, then we will have oscillations.[4] Whether the origin is stable or unstable will depend on the real part of $\lambda_{1,2}$. Because of the oscillations, we now refer to the origin as a **spiral**. If

4: Remember that, because of Euler's formula, we can write complex exponentials as sums of sines and cosines.

the eigenvalues are purely imaginary, the trajectories will oscillate the origin without going to or away from it, and we say that the origin is a (neutrally stable) **center** (recall example 11.1.1!).

Example 11.4.1 Study the behavior of the system

$$\begin{cases} \dot{x} = x - 4y \\ \dot{y} = x + y \end{cases}.$$

The matrix $\begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $\lambda_{1,2} = 1 \pm 2i$ and corresponding eigenvectors $\mathbf{v}_{1,2} = (\pm 2i, 1)$ (it is not hard to show that complex eigenvalues and eigenvectors always come in pairs of conjugates when the matrix A is real). Then the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(1+2i)t} \begin{pmatrix} 2i \\ 1 \end{pmatrix} + c_2 e^{(1-2i)t} \begin{pmatrix} -2i \\ 1 \end{pmatrix}.$$

But wait a minute! The original system was real! How come we get complex solutions! Well, not so fast. Let's apply Euler's formula and see what happens. Since $e^{(1+2i)t} = e^t(\cos 2t + i \sin 2t)$, we have:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^t \begin{pmatrix} -2\sin 2t + 2i\cos 2t \\ \cos 2t + i \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} -2\sin 2t - 2i\cos 2t \\ \cos 2t - i \sin 2t \end{pmatrix},$$

and note that the two vectors are conjugate! (this always happens, by the way). With a bit of algebra, we obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = (c_1 + c_2) e^t \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + i(c_1 - c_2) e^t \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix},$$

and the two vectors are the real and imaginary parts of $e^{(1+2i)t}\mathbf{v}_1$. Since $c_1 + c_2$ and $i(c_1 - c_2)$ are arbitrary constants that will have to be determined by the initial conditions (in this case $i(c_1 - c_2) = x_0$ and $C_1 + c_2 = y_0$), we can simply call them k_1 and k_2 and write

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^t \begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + k_2 e^t \begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix},$$

and there are no imaginary numbers any longer.

Note that this system will oscillate around the origin, getting farther and farther away from it (Figure 11.3).

In short, if the matrix A has complex eigenvalue λ and corresponding eigenvector $\mathbf{v}$ (no need to consider their conjugates), the general solution becomes

$$\mathbf{x}(t) = c_1 \mathrm{Re}\,(e^{\lambda t}\mathbf{v}) + c_2 \mathrm{Im}\,(e^{\lambda t}\mathbf{v}).$$

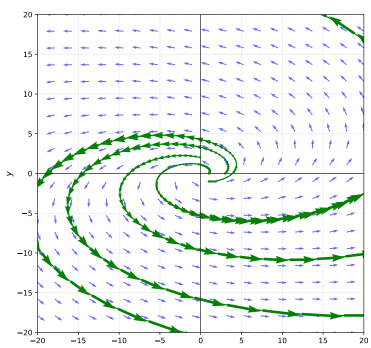


Figure 11.3: Vector field for the system in Example 11.4.1, some trajectories in green.

Exercises

Exercise 11.1 For the following systems, find (when possible) the general solution, plot the phase portrait and classify the fixed point. If the eigenvectors are real, indicate them:

1. $\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 3y \end{cases}$
2. $\begin{cases} \dot{x} = 3x - 4y \\ \dot{y} = x - y \end{cases}$
3. $\begin{cases} \dot{x} = 5x + 2y \\ \dot{y} = -17x - 5y \end{cases}$
4. $\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 8x - 6y \end{cases}$
5. $\begin{cases} \dot{x} = 5x + 10y \\ \dot{y} = -x - y \end{cases}$
6. $\begin{cases} \dot{x} = -3x + 2y \\ \dot{y} = x - 2y \end{cases}$
7. $\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = -2x + 3y \end{cases}$
8. $\begin{cases} \dot{x} = y \\ \dot{y} = -x - 2y \end{cases}$

Exercise 11.2 (Love Affairs, by Steven Strogatz) Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him and grows cold when she hates him.

Let

$$R(t) = \text{Romeo's love/hate for Juliet at time } t$$

$$J(t) = \text{Juliet's love/hate for Romeo at time } t.$$

Positive values of R and J signify love, while negative values signify hate. Then a model for their star-crossed romance is

$$\begin{cases} \frac{dR}{dt} = aJ \\ \frac{dJ}{dt} = -bR \end{cases}$$

where the parameters a and b are positive, to be consistent with the story.

Study the outcome of the system: will Romeo and Juliet find love?

Exercise 11.3 Now consider the forecast for lovers governed by the general linear system

$$\begin{cases} \frac{dR}{dt} = aR + bJ \\ \frac{dJ}{dt} = cR + dJ \end{cases}$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles.

1. What happens if $a = d < 0$ and $b = c > 0$?
2. What if $a = 0, b = 1, c = -1$ and $d = 1$? Classify the fixed point at the origin. Sketch $R(t)$ and $J(t)$ if $R(0) = 1, J(0) = 0$.
3. Suppose Romeo and Juliet react to each other, but not to themselves ($a = d = 0, b, c > 0$). What happens?

Exercise 11.4 A drug is administered to a person in a single dose. We assume that the drug does not accumulate in body tissue, but is filtered from the blood by the kidneys which then pass the drug into the urine.

We denote the amount of drug in the body at time t by $x_1(t)$ and in the urine at time t by $x_2(t)$. Initially, $x_1(0) = K$ and $x_2(0) = 0$. Suppose a fraction k_1 of the drug is filtered out by the kidneys in each unit of time. Then the movement of the drug between the body and the urine is modeled by

$$\begin{cases} \dot{x}_1 = -k_1 x_1 \\ \dot{x}_2 = k_1 x_1 \end{cases}$$

Solve for $x_1(t)$ and $x_2(t)$.

Exercise 11.5 Write the general solution of the harmonic oscillator

$$m\ddot{x} + kx = 0,$$

seen in Example 11.1.1. Find the particular solution if $x(0) = 0, \dot{x}(0) = 1$.

Exercise 11.6 Disturbances in forests (wind, fire, etc.) create gaps by killing trees. These gaps are eventually filled by new trees. We will model this process by a two-compartment model. We denote by $x_1(t)$ the area occupied by gaps and by $x_2(t)$ the area occupied by adult trees. We assume that the dynamics are given by

$$\begin{cases} \dot{x}_1 = -0.2x_1 + 0.1x_2 \\ \dot{x}_2 = 0.2x_1 - 0.1x_2 \end{cases}$$

1. Show that $x_1(t) + x_2(t)$ is a constant. Denote the constant by A and give its meaning.
2. Let $x_1(0) + x_2(0) = 20$. Use this to replace $x_2(t)$ therefore reducing the system to one variable $\dot{x}_1 = 2 - 0.3x_1$.
3. Solve the system and determine what fraction of the forest is occupied by adult trees at time t when $x_1(0) = 2$ and $x_2(0) = 18$. What happens as $t \longrightarrow \infty$?

Exercise 11.7 What happens to the system $\dot{\mathbf{x}} = A\mathbf{x}$ if the matrix A has two equal eigenvalues? Study the evolution of the system by sketching the phase portrait:

1. $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$
2. $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

12 Systems of Nonlinear Differential Equations

12.1 Introduction

Generally, we are interested in systems of differential equations of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, \dots, x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, \dots, x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, \dots, x_n)
\end{aligned} \tag{12.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1, 2, \dots, n$. We assume that the functions f_i, $i = 1, 2, \dots, n$, do not explicitly depend on t; this system is therefore called *autonomous*.

We no longer assume that the functions f_i are linear, as in Chapter 11. Using vector notation, we can write this system in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and $\mathbf{f}(\mathbf{x})$ is a vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ with components $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, n$. The function $\mathbf{f}(\mathbf{x})$ defines a vector field, just as in the linear case.

Unless the functions f_i are linear, it is typically not possible to find explicit solutions of systems of differential equations. If we want to solve such systems, we frequently must use numerical methods. Instead of trying to find solutions, we will focus on fixed points and their stability.

Just like for linear systems, we say that a point $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a **fixed point** (also called **critical point** or **equilibrium**) of the equation $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$, if

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) = \mathbf{0}.$$

This implies that if we start the solution of a system of differential equations at an equilibrium point, it will stay there for all later times.

As in the linear case, a solution might not return to an equilibrium after a small perturbation; if the solution returns to the equilibrium, we call it *stable*, while if the solution does not return, then we call the equilibrium *unstable*. In the next section we will see how to analyze the stability of a fixed point analytically. We will restrict our discussion to systems of two equations in two variables.[1]

1: The concepts are the same when we have more than two equations, but the calculations become more involved.

12.2 Stability of fixed points

For any fixed point $\mathbf{x}^*$ that satifies $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, we look at what happens to a small perturbation to determine its stability. That is, we look at how $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\eta}$ changes under the dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ assuming that $\boldsymbol{\eta}$ is very small:

$$\frac{d}{dt}(\mathbf{x}^* + \boldsymbol{\eta}) = \frac{d\boldsymbol{\eta}}{dt} = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\eta}).$$

The linearization of $\mathbf{f}(\mathbf{x})$ about $\mathbf{x}^*$ is

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\eta} = D\mathbf{f}(\mathbf{x})\boldsymbol{\eta},$$

since $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. Here $D\mathbf{f}(\mathbf{x}^*)$ is the Jacobian matrix evaluated at $\mathbf{x}^*$:

Definition 12.2.1 (Jacobian matrix, 2×2 case) *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The Jacobian matrix of f is the $m \times n$ matrix of all partial derivatives of f, given by*

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix},$$

*where $\frac{\partial f_1}{\partial x}$ is the **partial derivative** of f_1 with respect to x, which is caclulated by taking y as a constant and computing the derivative of f_1 as a function of x only (and similarly with the remaining partial derivatives). The Jacobian matrix is sometimes denoted by $J_{\mathbf{f}}(\mathbf{x})$.*

Example 12.2.1

1. If $f(x, y) = x^2y + y^3$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

   To find $\frac{\partial f}{\partial x}$ we hold y constant and differentiate only with respect to x; this yields

   $$\frac{\partial}{\partial x}(x^2y + y^3) = \frac{\partial f}{\partial x} = 2xy.$$

   Similarly, to find $\frac{\partial f}{\partial y}$ we hold x constant and differentiate only with respect to y:

   $$\frac{\partial}{\partial y}(x^2y + y^3) = \frac{\partial f}{\partial y} = x^2 + 3y^2.$$

2. Find $\frac{\partial f}{\partial x}$ if $f(x, y) = \frac{xy}{x^2+y^2}$. By the quotient rule,

   $$\frac{\partial f}{\partial x} = \frac{y(x^2 + y^2) - 2x^2y}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}.$$

Our previous discussion means that we can approximate $\mathbf{f}(\mathbf{x} + \boldsymbol{\eta})$ by its linearization $D\mathbf{f}(\mathbf{x})\boldsymbol{\eta}$, leading to

$$\frac{d\boldsymbol{\eta}}{dt} = D\mathbf{f}(\mathbf{x})\boldsymbol{\eta},$$

which is the linear approximation of the dynamics of the perturbation $\boldsymbol{\eta}$.

Note that $D\mathbf{f}(\mathbf{x})$ is a 2×2 matrix of constants, so we have a linear system of equations. The eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}^*)$ allow us to determine the nature of the equilibrium, just like in Section 11.4. This is a local analysis, just as in the case of a single differential equation, since the linearization is a good approximation only as long as we are sufficiently close to the point about which we linearized. In the limit cases, where $\lambda_1 = \lambda_2$ or if λ is purely imaginary, or if one of the eigenvalues is zero, the behavior may differ from the linearized case. In these cases, the nonlinear terms cannot reasonably be neglected. There are additional methods that one can use to analyze the stability of equilibria from boundary regions, but these methods are beyond the scope of this course. You need to know that linearization cannot be trusted for these equilibria, and you may regard them as being unclassifiable for the time being.

The main challenge when identifying equilibria in nonlinear equations is that we must solve a system of equations to find all the points where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. These equations will, in general, be nonlinear, so we cannot solve them using general methods. Typically, we must use one of the equations to eliminate a variable; that is, we must rewrite the other equation in terms of a single variable. We may then solve the rewritten equation in a single variable. Let's see an example:

Example 12.2.2 Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x - 2x^2 - 2xy \\ \frac{dy}{dt} &= 4y - 5y^2 - 7xy. \end{aligned} \tag{12.2}$$

To find equilibria, we set the right-hand sides of (12.2) equal to zero:

$$\begin{aligned} x - 2x^2 - 2xy = 0 &\implies x(1 - 2x - 2y) = 0, \\ 4y - 5y^2 - 7xy = 0 &\implies y(4 - 5y - 7x) = 0. \end{aligned}$$

From the first equation, either $x = 0$ or $2x + 2y = 1$. Using these cases:

- If $x = 0$, the second equation implies $y(4 - 5y) = 0$, so $y = 0$ or $y = \frac{4}{5}$.
- If $2x + 2y = 1$ or, equivalently, $x = \frac{1}{2} - y$, we get:

$$y\left(4 - 5y - 7\left(\frac{1}{2} - y\right)\right) = 0 \implies y\left(\frac{1}{2} + 2y\right) = 0,$$

so $y = 0$ or $y = -\frac{1}{4}$. Substituting into $x = \frac{1}{2} - y$, we find:

$$\text{If } y = 0,\ x = \frac{1}{2}; \quad \text{if } y = -\frac{1}{4},\ x = \frac{3}{4}.$$

To summarize, there are four equilibria: $(0, 0)$, $\left(0, \frac{4}{5}\right)$, $\left(\frac{1}{2}, 0\right)$, $\left(\frac{3}{4}, -\frac{1}{4}\right)$.

To analyze stability, we compute the Jacobian matrix:

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 - 4x - 2y & -2x \\ -7y & 4 - 10y - 7x \end{pmatrix}.$$

Now, evaluate $D\mathbf{f}$ at each equilibrium:

- At $(0,0)$:
$$D\mathbf{f}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$
The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$. Both are positive, so $(0,0)$ is an unstable node.
- At $\left(0, \frac{4}{5}\right)$:
$$D\mathbf{f}\left(0, \frac{4}{5}\right) = \begin{pmatrix} -\frac{3}{5} & 0 \\ -\frac{28}{5} & -4 \end{pmatrix}.$$
The eigenvalues are $\lambda_1 = -\frac{3}{5}$ and $\lambda_2 = -4$. Both are negative, so $\left(0, \frac{4}{5}\right)$ is a stable node.
- At $\left(\frac{1}{2}, 0\right)$:
$$D\mathbf{f}\left(\frac{1}{2}, 0\right) = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$
The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$. One is positive and one is negative, so $\left(\frac{1}{2}, 0\right)$ is a saddle point.
- At $\left(\frac{3}{4}, -\frac{1}{4}\right)$:
$$D\mathbf{f}\left(\frac{3}{4}, -\frac{1}{4}\right) = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{7}{4} & \frac{5}{4} \end{pmatrix}.$$
The eigenvalues are complex with negative real parts, so $\left(\frac{3}{4}, -\frac{1}{4}\right)$ is a stable spiral.

12.3 Graphical Analysis of Nonlinear Systems

We have shown how to use linearization to understand how solutions behave near the point equilibria of a system of nonlinear equations. What other information can be gleaned from the system? In this section, we describe a graphical method for analyzing the behavior of solutions over the entire plane.

If $\dot{x} = f_1(x,y)$ and $\dot{y} = f_2(x,y)$, the curves

$$f_1(x,y) = 0 \quad \text{and} \quad f_2(x,y) = 0$$

are called *zero isoclines* or **nullclines**, and they represent the points in the x–y plane where either $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$. The point where both nullclines intersect is a fixed point, and we can study its stability using linearization.

On the $f_1 = 0$ isoclines, $\frac{dx}{dt} = 0$, so the direction vectors must point either vertically upward (if $f_2 > 0$) or vertically downward (if $f_2 < 0$). Similarly, on the $f_2 = 0$ isocline, $\frac{dy}{dt} = 0$, so the direction vectors must point horizontally, either to the right (if $f_1 > 0$) or to the left (if $f_1 < 0$). Here is an important observation: if $f_1 > 0$ at one point of an $f_2 = 0$ isocline, then $f_1 > 0$ along the entire isocline until we reach an equilibrium. This is because f_1 can only change its sign at a point where $f_1 = 0$, and if $f_1 = 0$ on the $f_2 = 0$ isocline, that point is an equilibrium. Similarly, if a segment of the $f_2 = 0$ isocline does not contain an equilibrium, f_1 must have the same sign over that entire segment. We can use similar arguments to plot the directions of the vector field in the regions between nullclines.

Example 12.3.1 The nullclines of the previous system (12.2) are

$$\frac{dx}{dt} = 0 \implies x = 0 \text{ or } y = \frac{1}{2} - x$$
$$\frac{dy}{dt} = 0 \implies y = 0 \text{ or } y = \frac{4}{7} - \frac{5}{7}x.$$

The nullclines, fixed points and some trajectories of the system are plotted in Figure **??**.

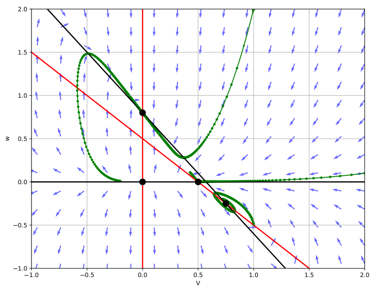


Figure 12.1: Fixed points, nullclines and trajectories of the system (12.2).

12.4 Fitzhugh–Nagumo Model of a Neuron

We can think of a neuron as a bistable system; that is, it can exist in one of two stable states: either with Na^+ ions outside and K^+ ions inside, or vice versa. The neuron transitions between these states only when it receives a sufficiently strong stimulus, making it an *excitable system*.

Fitzhugh (1961) and Nagumo et al. (1962) developed a model that captures these dynamics, characterized by two variables: V, the voltage difference across the neuron membrane, representing the net difference in charge inside and outside the cell, and w, modeling the sodium and potassium ion channels that regulate ion flow. The Fitzhugh–Nagumo equations are given by:

$$\begin{aligned} \frac{dV}{dt} &= -V(V-a)(V-1) - w, \\ \frac{dw}{dt} &= V - cw, \end{aligned} \tag{12.3}$$

where a and c are constants satisfying $0 < a < 1$ and $c > 0$.

The nullclines of the system are

$$w = -V(V-a)(V-1) \quad \text{and} \quad w = \frac{V}{c}.$$

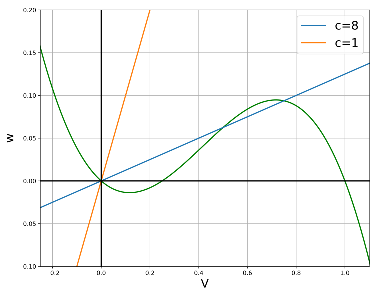


Figure 12.2: Nullclines of the Fitzhugh-Nagumo model with $c = 8$ (blue) and $c = 1$ (orange).

The zero isoclines are shown in Figure 12.2, where the $dV/dt = 0$ isocline is a cubic curve, and the $dw/dt = 0$ isocline is a straight line. The behavior of the system depends on the parameter c: for small c, the isoclines intersect only once at $(0, 0)$, while for larger c, the isoclines intersect three times, resulting in three equilibria.

To analyze stability, we linearize the system around each equilibrium. The Jacobian matrix is:

$$D\mathbf{f}(V, w) = \begin{pmatrix} -3V^2 + 2V + +2aV - a & -1 \\ 1 & -c \end{pmatrix}.$$

At $(0, 0)$:

$$D\mathbf{f}(0,0) = \begin{pmatrix} -a & -1 \\ 1 & -c \end{pmatrix},$$

with both eigenvalues with a negative real part. Therefore, $(0, 0)$ is always a stable equilibrium.

But what happens for large values of c? The three equilibria obey

$$-V(V-a)(V-1) = w, \qquad w = V/c \implies -V(V-a)(V-1) = \frac{V}{c},$$

so either $V = 0$ or $c(V-a)(V-1) = -1$. The last equation is solvable, but let's see one particular example.

Proposed Exercise 12.4.1 Find the three equilibria of the model when $a = 1/4$ and $c = 8$.

For $a = 1/4$ and $c = 8$, the equilibria are $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{16}\right)$, $\left(\frac{3}{4}, \frac{3}{32}\right)$.

Let's look at the stability of the new equilibria: at $\left(\frac{1}{2}, \frac{1}{16}\right)$, the Jacobian matrix is:

$$D\mathbf{f}\left(\frac{1}{2}, \frac{1}{16}\right) = \begin{pmatrix} -\frac{1}{4} & -1 \\ 1 & -8 \end{pmatrix}.$$

Here, $\det(D\mathbf{f}) < 0$, so $\left(\frac{1}{2}, \frac{1}{16}\right)$ is a saddle point.

At $\left(\frac{3}{4}, \frac{3}{32}\right)$, the Jacobian matrix is:

$$D\mathbf{f}\left(\frac{3}{4}, \frac{3}{32}\right) = \begin{pmatrix} -\frac{1}{16} & -1 \\ 1 & -8 \end{pmatrix}.$$

Here, $\det(D\mathbf{f}) > 0$ and $\operatorname{tr}(D\mathbf{f}) < 0$, so $\left(\frac{3}{4}, \frac{3}{32}\right)$ is a stable node.

For small c, the neuron always returns to $(0, 0)$, representing the resting state. For larger c, the neuron exhibits bistability, with two stable equilibria at $(0, 0)$ and $\left(\frac{3}{4}, \frac{3}{32}\right)$. The system's behavior depends on the initial conditions:

- If $V(0) < V_c$, the neuron returns to $(0, 0)$ (resting state).
- If $V(0) > V_c$, the neuron fires and converges to $\left(\frac{3}{4}, \frac{3}{32}\right)$.

Figure 12.3 illustrates the solution curves and potential $V(t)$ for different initial conditions. For weak stimuli ($V(0) < V_c$), $V(t)$ decays to 0. For stronger stimuli ($V(0) > V_c$), $V(t)$ converges to the fired state.

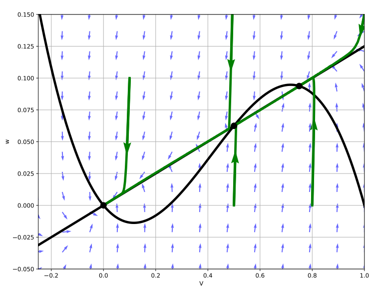


Figure 12.3: Vector field for the Fitzhugh-Nagumo model (Eq. (12.3)) with the nullclines and some trajectories in green. Parameters: $a = 1/4$, $c = 8$.

Exercises

Exercise 12.1 Find all fixed points of each system of differential equations and determine their stability.

1. $\begin{cases} \frac{dx}{dt} &= -x + 2x(1-x), \\ \frac{dy}{dt} &= -y + 5y(1-x-y). \end{cases}$
2. $\begin{cases} \frac{dx}{dt} &= 2x - x^2 - 2yx, \\ \frac{dy}{dt} &= y - 2y^2 - xy. \end{cases}$
3. $\begin{cases} \frac{dx}{dt} &= 4x(1-x) - 2xy, \\ \frac{dy}{dt} &= y(2-y) - y. \end{cases}$
4. $\begin{cases} \frac{dx}{dt} &= xy - 2y, \\ \frac{dy}{dt} &= x + y. \end{cases}$

Exercise 12.2 Assume that $a > 0$. Find all point equilibria of the following system of differential equations and characterize their stability:

$$\begin{cases} \frac{dx}{dt} &= y(x-a), \\ \frac{dy}{dt} &= y^2 - x. \end{cases}$$

Exercise 12.3 Consider the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} &= x(10 - 2x - y), \\ \frac{dy}{dt} &= y(10 - x - 2y). \end{cases}$$

(a) Graph the zero isoclines.
(b) Find all equilibria and classify them by linearizing the system near each equilibrium.
(c) Draw the directions of the vector field on the zero isoclines and in the regions between the zero isoclines.

Exercise 12.4 The Lotka-Volterra model of interspecific competition for two species is given by the following equations:

$$\begin{cases} \frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} - \frac{\alpha_{12} N_2}{K_1}\right), \\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - \frac{N_2}{K_2} - \frac{\alpha_{21} N_1}{K_2}\right). \end{cases}$$

The coefficients $r_1, r_2, K_1, K_2, \alpha_{12}, \alpha_{21}$ are all positive. Take $r_1 = 1, r_2 = 1, K_1 = 1, K_2 = 1$.

1. Find the fixed points and study their stability if $a_{12} = 0.4, a_{21} = 2$.
2. Find the fixed points and study their stability if $a_{12} = 2, a_{21} = 0.4$.
3. Find the fixed points and study their stability if $a_{12} = 0.4, a_{21} = 0.4$.
4. Find the fixed points and study their stability if $a_{12} = 2, a_{21} = 2$.

Exercise 12.5 The Lotka-Volterra model for predator-prey interactions is

$$\begin{cases} \frac{dN}{dt} &= rN - aPN, \\ \frac{dP}{dt} &= bPN - dP, \end{cases}$$

where r is the per capita growth rate of N, the prey, d is the death rate of the predator P, and a, b measure how interactions affect the densities of N and P respectively.

If $r = 5, a = b = d = 1$,

(a) Show that this system has two equilibria: the trivial equilibrium $(0, 0)$, and a nontrivial one in which both species have positive densities.
(b) Use the eigenvalue approach to show that the trivial equilibrium is unstable.
(c) Determine the eigenvalues corresponding to the nontrivial equilibrium. Does your analysis allow you to infer anything about the stability of this equilibrium?

Exercise 12.6 Assume the following example of the FitzHugh–Nagumo model:

$$\begin{cases} \frac{dV}{dt} & = -V(V - 3/5)(V - 1) - w, \\ \frac{dw}{dt} & = V - cw. \end{cases}$$

Find the smallest value of c for which the model predicts the existence of multiple equilibria.

APPENDIX

Solutions to Exercises A

A.1 Functions

Exercise 1.1

(i) We can factor out the denominator as $x^2 - 5x + 6 = (x - 2)(x - 3)$; therefore, the domain is $\mathbb{R} - \{2, 3\}$.

(ii) There are two conditions for $f(x)$ to exist: $1 - x^2 \geq 0$ and $x^2 - 1 \geq 0$. Together they imply $1 - x^2 = 0$. Therefore the domain is just the set $\{-1, 1\}$.

(iii) There are two conditions to be met for x to be in the domain: first, $1 - x^2 \geq 0$; second, $x \neq \sqrt{1 - x^2}$. The first condition implies $x^2 \leq 1$, or equivalently, $-1 \leq x \leq 1$. The second condition is not fulfilled if $x = \sqrt{1 - x^2}$. Squaring this equation we obtain $x^2 = 1 - x^2$, which is equivalent to $x^2 = 1/2$. The two solutions of this equation are $x = \pm 1/\sqrt{2}$, but of them two, only the positive one is a solution of the original equation $x = \sqrt{1 - x^2}$. Thus the domain is $[-1, 1/\sqrt{2}) \cup (1/\sqrt{2}, 1]$.

(iv) The two coditions to be met for x to be in the domain are $4 - x^2 \geq 0$ and $1 - \sqrt{4 - x^2} \geq 0$. The first one reads $x^2 \leq 4$, i.e., $-2 \leq x \leq 2$. The second one implies $\sqrt{4 - x^2} \leq 1$. Both sides of this inequality are positive, so we can square it to obtain $4 - x^2 \leq 1$, i.e., $x^2 \geq 3$. This holds either if $x \geq \sqrt{3}$ or $x \leq -\sqrt{3}$. Therefore, the domain is $[-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$.

(v) The denominator vanishes if $\log x = 1$, i.e., if $x = e$. Since the logarithm requires $x > 0$, the domain is $(0, e) \cup (e, \infty)$.

(vi) The condition to be met now is $x - x^2 > 0$. We can factor $x - x^2 = x(1 - x)$, so the roots of the parabola are $x = 0$ and $x = 1$. Since the coefficient of x^2 is negative, the parabola is positive provided $0 < x < 1$. The domain is then $(0, 1)$.

(vii) Three conditions need to be met: first, $x > 0$ because x is the argument of a logarithm; second, $\log x \neq 0$ because it is the denominator; and third, $5 - x \geq 0$ because it is the argument of a square root. The second condition implies $x \neq 1$, whereas the third one implies $x \leq 5$. Thus the domain is $(0, 1) \cup (1, 5]$.

Exercise 1.2

(a) We know that $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Then

$$(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(-x) = -(f + g)(x),$$

so $f + g$ is *odd*. Now,

$$(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x),$$

so fg is *even*. Finally,

$$(f \circ g)(-x) = f\big(g(-x)\big) = f\big(-g(x)\big) = -f\big(g(x)\big) = -(f \circ g)(x).$$

Thus $f \circ g$ is *odd*.

(b) Now $f(-x) = f(x)$ and $g(-x) = -g(x)$. Then

$$(f + g)(-x) = f(-x) + g(-x) = f(x) - g(-x),$$

so $f + g$ is neither even nor odd. As for the product,

$$(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -f(x)g(x) = -(fg)(x),$$

so fg is *odd*. Finally,

$$(f \circ g)(-x) = f\big(g(-x)\big) = f\big(-g(x)\big) = f\big(g(x)\big) = (f \circ g)(x).$$

Thus $f \circ g$ is *even*.

Exercise 1.3 (i)

$$f(-x) = \frac{-x}{(-x)^2 + 1} = -f(x).$$

The function is *odd*.

(ii)

$$f(-x) = \frac{(-x)^2 - (-x)}{(-x)^2 + 1} = \frac{x^2 + x}{x^2 + 1} \neq \pm f(x),$$

so the function is neither.

(iii)

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x} = f(x).$$

The function is *even*.

(iv)

$$f(-x) = \cos\left((-x)^3\right)\sin\left((-x)^2\right)e^{-(-x)^4} = \cos(-x^3)\sin(x^2)e^{-x^4} = \cos(x^3)\sin(x^2)e^{-x^4} = f(x).$$

The function is *even*.

(v)

$$f(-x) = \frac{1}{\sqrt{(-x)^2 + 1} - (-x)} = \frac{1}{\sqrt{x^2 + 1} + x},$$

so the function is neither.

(vi) This function is the logarithm of the function in the previous item, so it seems that it has no defined parity because

$$f(-x) = \log\left(\sqrt{x^2 + 1} + x\right).$$

However,

$$\sqrt{x^2 + 1} + x = \frac{\left(\sqrt{x^2 + 1} + x\right)\left(\sqrt{x^2 + 1} - x\right)}{\sqrt{x^2 + 1} - x} = \frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} - x} = \frac{1}{\sqrt{x^2 + 1} - x},$$

so

$$f(-x) = \log\left(\sqrt{x^2+1} + x\right) = \log\left(\frac{1}{\sqrt{x^2+1} - x}\right) = -\log\left(\sqrt{x^2+1} - x\right) = -f(x).$$

The function is *odd*.

Exercise 1.4

(a) An easy way to check for injectivity is to determine whether the equation $y = f(x)$ has a unique solution for those y for which it can be solved.

(i) For every $y \in \mathbb{R}$,

$$y = 7x - 4 \quad \Rightarrow \quad x = \frac{y+4}{7}.$$

So there is a unique solution no matter y, which means that the function is injective.

(ii) Only if $-1 \le y \le 1$ the equation

$$y = \sin(7x - 4)$$

can have a solution. On the other hand, two points x_1 and x_2 such that $7x_2 - 4 = 7x_1 - 4 + 2n\pi$, with $n \in \mathbb{Z}$, are both solutions of the same y. Clearly $x_2 = x_1 + 2n\pi/7$. Therefore there are infinitely many solutions for each $-1 \le y \le 1$, which means that the function is not injective.

(iii) For any $y \in \mathbb{R}$,

$$y = (x+1)^3 + 2 \quad \Rightarrow \quad x = (y-2)^{1/3} - 1,$$

so the solution is unique and the function is injective.

(iv) Take y so that

$$y = \frac{x+2}{x+1}.$$

Then

$$y(x+1) = x + 2 \quad \Rightarrow \quad y - 2 = x(1-y).$$

Thus, provided $y \ne 1$, we obtain

$$x = \frac{y-2}{1-y}$$

and the solution is unique. The function is injective.

(v) Take y and solve for $y = x^2 - 3x + 2$, or $x^2 - 3x + 2 - y = 0$. Then

$$x = \frac{3 \pm \sqrt{9 + 4(y-2)}}{2} = \frac{3 \pm \sqrt{4y+1}}{2}.$$

The equation has a solution only if $y \ge -1/4$. But for all $y > -1/4$ there are two different solutions. Therefore the function is not injective.

(vi) Consider the equation

$$y = \frac{x}{x^2+1}.$$

If $y = 0$ the only solution is $x = 0$. If $y \neq 0$ it can be transformed into

$$y(x^2+1) = x \quad \Rightarrow \quad yx^2 - x + y = 0.$$

The solutions of this quadratic equation are

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}.$$

There is solution only if $y^2 \leq 1/4$, i.e., $-1/2 \leq y \leq 1/2$, but for every $-1/2 < y < 1/2$ there are two different solutions for the same y, hence the function is not injective.

(vii) For every $y > 0$,

$$y = e^{-x} \quad \Rightarrow \quad \log y = -x \quad \Rightarrow \quad x = -\log y.$$

The solution is unique and the function is injective.

(viii) For every $y \in \mathbb{R}$,

$$y = \log(x+1) \quad \Rightarrow \quad e^y = x+1 \quad \Rightarrow \quad x = e^y - 1.$$

The solution is unique and the function is injective.

(b) The solutions of the equation $y = x^2 - 3x + 2$ are (see previous item)

$$x = \frac{3 \pm \sqrt{4y+1}}{2}.$$

Clearly one solution is larger than $3/2$ and the other is smaller than $3/2$. Therefore, if we limit the domain to those x larger than $3/2$ only one solution survives and the function becomes injective.

(c)

(i) There is a unique solution for every $y \in \mathbb{R}$, therefore the function is surjective, hence bijective.

(ii) Not surjective because the range is $[-1, 1]$.

(iii) Surjective and bijective.

(iv) Not surjective because $y = 1$ is not in the range of the function.

(v) Not surjective because the range is $[-1/4, \infty)$.

(vi) Not surjective because the range is $[-1/2, 1/2]$.

(vii) Not surjective because the range is $(0, \infty)$.

(viii) Surjective and bijective.

Exercise 1.5 1. Since $\sin(2x - \pi) \in [-1, 1]$,

$$\min f(x) = -3+1 = -2, \quad \max f(x) = 3+1 = 4, \quad A = \max f(x) - \min f(x) = 4-(-2) = 6.$$

2. The inner argument $2x - \pi$ has period 2π. Hence

$$2(x+c) - \pi = 2x - \pi + 2\pi \quad \Longrightarrow \quad c = \pi.$$

3. Write
$$2x - \pi = 2\left(x - \tfrac{\pi}{2}\right),$$
so the graph is shifted to the right by $\frac{\pi}{2}$.
4. Compared to $3\sin 2x$, the graph is moved up by 1.

Exercise 1.6 1.
$$\sin\left(x + \tfrac{\pi}{2}\right) = \sin x \cos \tfrac{\pi}{2} + \cos x \sin \tfrac{\pi}{2} = \cos x.$$

2.
$$\cos\left(x - \tfrac{\pi}{2}\right) = \cos x \cos \tfrac{\pi}{2} + \sin x \sin \tfrac{\pi}{2} = \sin x.$$
Hence $\sin x$ and $\cos x$ differ by a phase-shift of $\frac{\pi}{2}$.
3. From the double-angle formula $\cos 2x = \cos(x+x) = \cos^2 x - \sin^2 x$, we get
$$\cos^2 x + \sin^2 x = 1.$$
4. Again from $\cos 2x = 2\cos^2 x - 1$, we solve for $\cos^2 x$:
$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$
5. Since $\sin^2 x = 1 - \cos^2 x$, it follows that
$$\sin^2 x = 1 - \frac{1 + \cos 2x}{2} = \frac{1 - \cos 2x}{2}.$$

Exercise 1.7
$$2\cos x - 3 = 0 \quad \Longrightarrow \quad \cos x = \tfrac{3}{2}.$$
Since $\frac{3}{2} \notin [-1, 1]$, there are no real solutions in $[0, 2\pi)$.

Exercise 1.8 Use GeoGebra to help you with this exercise.

Exercise 1.9 Here are some hints to help you plot these functions:

(i) Start off with the plot of $g(x) = x^2$; function $f(x) = g(x+2) - 1$, so shift the plot two units to the left and one unit down.

(ii) Start off with the plot of $g(x) = \sqrt{x}$ and then tranform it into that of $h(x) = \sqrt{-x}$ by reflecting it on the Y axis. Then $f(x) = h(x - 4)$, so shift this plot four units to the right.

(iii) Start off from the plots of $g_1(x) = x^2$ and $g_2(x) = 1/x$. Near $x = 0$ g_1 is negligible with respect to g_2 —which diverges to $\pm\infty$ at $x = 0$. Far from $x = 0$ it is g_2 that is negligible with respect to g_1, which grows indefinitely. So $f(x)$ is close to $g_2(x)$ as x 'moves' toward 0, and close to $g_1(x)$ as x goes far awat from $x = 0$. Sketch the plot of $f(x)$ using this information.

(iv) Start off with the plot of $g(x) = x^2$ and shift it up one unit to get that of $h(x) = x^2 + 1$. Then $f(x) = 1/h(x)$. Since $h(x) > 1$ for all $x \neq 0$ and $h(0) = 1$, then $f(x) < 1$ for all $x \neq 0$ and $f(0) = 1$. Besides, $h(x)$ grows indefinitely as x goes away from the origin, so $f(x)$ has to approach 0.

(v) $g(x) = x - x^2 = x(1 - x)$, so $g(x) > 0$ if $0 < x < 1$ and $g(x) < 0$ if

$x < 0$ or $x > 1$. Therefore

$$f(x) = \begin{cases} x^2, & \text{if } 0 \le x \le 1, \\ x, & \text{otherwise.} \end{cases}$$

(vi) e^x is monotonically increasing and croses 1 at $x = 0$. Therefore

$$f(x) = \begin{cases} e^x - 1, & \text{if } x \ge 0, \\ 1 - e^x, & \text{if } x < 0. \end{cases}$$

All that needs to be done is to reflect the graph of $e^x - 1$ (equal to that of e^x but shifted down one unit) for $x < 0$ on the X axis.
Let n be an integer and let us try to figure out where

$$\left\lfloor \frac{1}{x} \right\rfloor = n.$$

By definition

$$n \le \frac{1}{x} < n + 1. \tag{A.1}$$

As we have mentioned above, $f(x)$ will not be defined if $n = 0$. This means all x such that

$$0 \le \frac{1}{x} < 1.$$

The left inequality implies $x > 0$. The right inequality implies $x > 1$. Therefore the domain of f is $(-\infty, 0) \cup (0, 1]$.
Consider first $x \in (0, 1]$. Then, according to (A.1) $n > 0$. From the left inequality $x \le 1/n$, and from the right one $x > 1/(n + 1)$. Thus

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \qquad n \in \mathbb{N}.$$

In other words, $f(x) = 1$ for $x \in (1/2, 1]$, $f(x) = 1/2$ for $x \in (1/3, 1/2]$, $f(x) = 1/3$ for $x \in (1/4, 1/3]$, etc. This covers the plot of $f(x)$ within the interval $(0, 1]$. By the way, the function gets closer and closer to 0 as x approaches 0.
Consider now the interval $(-\infty, 0)$. Then n in (A.1) must be negative. Then the left inequality again implies $x \le 1/n$ and the right one $x > 1/(n + 1)$. The result is the same:

$$f(x) = \frac{1}{n} \quad \text{for all } x \in \left(\frac{1}{n+1}, \frac{1}{n} \right], \qquad n \in -\mathbb{N}.$$

So we have $f(x) = -1$ if $x \in (-\infty, -1]$, $f(x) = -1/2$ if $x \in (-1, -1/2]$, $f(x) = -1/3$ if $x \in (-1/2, -1/3]$, etc. This covers the whole interval $(-\infty, 0)$.

(vii) Function $g(x) = x^2 - 1 < 0$ if $-1 < x < 1$ and $g(x) > 0$ otherwise, so

$$f(x) = \begin{cases} 1 - x^2, & \text{if } -1 < x < 1, \\ x^2 - 1, & \text{otherwise.} \end{cases}$$

All that one has to do is to reflect the portion of the graph of $x^2 - 1$ in the interval $(-1, 1)$ on the X axis.

(viii) Plot $g(x) = e^x$. The plot of $g(-x)$ is just the mirror image with respect to the Y axis. And that of $-g(-x)$ is a new reflection with respect to the X axis. Shift the whole plot one unit upward and you will get the plot of $f(x) = -g(-x) + 1 = 1 - e^{-x}$.

(ix) The function is defined only if $|x| \geq 1$. Besides, it is an even function, so it will be symmetric with respect to the Y axis. Let us then focus on the positive interval $[1, \infty)$. Notice that $f(x) = \log(x-1)+\log(x+1)$. These are two graphs of $\log x$, the first one shifted one unit to the right and the second one shifted one unit to the left. Since $\log x$ grows very slowly but diverges at $x = 0$, near the point $x = 1$ function $\log(x-1)$ will diverge and $\log(x+1)$ will then be negligible. In oher words, $f(x) \approx \log(x-1)$. On the other hand, when x is large $x \pm 1 \approx x$, so $f(x) \approx 2\log x$. Plot $f(x)$ using this information.

(x) As x grows far away from the origin (positive or negative) $1/x$ becomes very small, so $\sin(1/x)$ approaches $1/x$, and therefore $f(x)$ approaches 1. On the other hand, $\sin(1/x)$ oscillates wildly as x gets near the origin, but x modulates the amplitude (making it smaller the closer to the origin).

Exercise 1.10

(i) We use the identity $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$ and obtain

$$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}} = na^{n-1}.$$

(ii) We use the identity $x - a = \left(\sqrt{x} - \sqrt{a}\right)\left(\sqrt{x} + \sqrt{a}\right)$ and get

$$\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} = \lim_{x\to a} \frac{\cancel{\sqrt{x} - \sqrt{a}}}{\cancel{\left(\sqrt{x} - \sqrt{a}\right)}\left(\sqrt{x} + \sqrt{a}\right)} = \frac{1}{2\sqrt{a}}.$$

(iii) We can rewrite

$$1-\sqrt{1-x^2} = \frac{\left(1 - \sqrt{1-x^2}\right)\left(1 + \sqrt{1-x^2}\right)}{1 + \sqrt{1-x^2}} = \frac{1 - (1-x^2)}{1 + \sqrt{1-x^2}} = \frac{x^2}{1 + \sqrt{1-x^2}}.$$

Therefore

$$\lim_{x\to 0} \frac{1 - \sqrt{1-x^2}}{x^2} = \lim_{x\to 0} \frac{\cancel{x^2}}{\cancel{x^2}\left(1 + \sqrt{1-x^2}\right)} = \lim_{x\to 0} \frac{1}{1 + \sqrt{1-x^2}} = \frac{1}{2}.$$

(iv) We can rewrite

$$\frac{1}{\sqrt{x}-1} = \frac{\sqrt{x}+1}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)} = \frac{\sqrt{x}+1}{x-1}.$$

Therefore

$$\lim_{x\to 1}\left(\frac{1}{\sqrt{x}-1} - \frac{2}{x-1}\right) = \lim_{x\to 1}\left(\frac{\sqrt{x}+1}{x-1} - \frac{2}{x-1}\right) = \lim_{x\to 1}\frac{\sqrt{x}+1-2}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$$
$$= \lim_{x\to 1}\frac{\cancel{x-1}}{\left(\sqrt{x}+1\right)\cancel{(x-1)}} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

Exercise 1.11

(i) On the one hand, as $x \longrightarrow \infty$,

$$x^3 + 4x - 7 = x^3 \left(1 + \frac{4}{x^2} - \frac{7}{x^3}\right) \sim x^3.$$

On the other hand,

$$7x^2 - \sqrt{2x^6 + x^5} = 7x^2 - x^3\sqrt{2 + \frac{1}{x}} = x^3 \left(\frac{7}{x} - \sqrt{2 + \frac{1}{x}}\right) \sim -\sqrt{2}x^3.$$

Therefore

$$\lim_{x\to\infty} \frac{x^3 + 4x - 7}{7x^2 - \sqrt{2x^6 + x^5}} = \lim_{x\to\infty} \frac{\cancel{x^3}}{-\sqrt{2}\cancel{x^3}} = -\frac{1}{\sqrt{2}}.$$

(ii) On the one hand, as $x \longrightarrow \infty$,

$$x + \sin x^3 = x \left(1 + \frac{\sin x^3}{x}\right) \sim x$$

because $|\sin x^3| \leq 1$ for all $x \in \mathbb{R}$. On the other hand,

$$5x + 6 \sim 5x.$$

Therefore

$$\lim_{x\to\infty} \frac{x + \sin x^3}{5x + 6} = \lim_{x\to\infty} \frac{\cancel{x}}{5\cancel{x}} = \frac{1}{5}.$$

(iii) As $x \longrightarrow \infty$,

$$\sqrt{x + \sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \frac{1}{x}\sqrt{x + \sqrt{x}}} = \sqrt{x}\sqrt{1 + \sqrt{\frac{1}{x} + \frac{1}{x^{3/2}}}} \sim \sqrt{x},$$

thus

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x + \sqrt{x + \sqrt{x}}}} = \lim_{x\to\infty} \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = 1.$$

(iv) This is an indeterminacy $\infty - \infty$, so we must transform

$$\sqrt{x^2 + 4x} - x = \frac{\left(\sqrt{x^2 + 4x} - x\right)\left(\sqrt{x^2 + 4x} + x\right)}{\sqrt{x^2 + 4x} + x} = \frac{x^2 + 4x - x^2}{\sqrt{x^2 + 4x} + x} = \frac{4x}{\sqrt{x^2 + 4x} + x}.$$

Now, as $x \longrightarrow \infty$,

$$\sqrt{x^2 + 4x} + x = x\left(\sqrt{1 + \frac{4}{x}} + 1\right) \sim 2x,$$

therefore

$$\lim_{x\to\infty} \left(\sqrt{x^2 + 4x} - x\right) = \lim_{x\to\infty} \frac{4x}{\sqrt{x^2 + 4x} + x} = \lim_{x\to\infty} \frac{4\cancel{x}}{2\cancel{x}} = 2.$$

Exercise 1.12

(i) Numerator and denominator are continuous functions in $\mathbb{R}$, so this function will be continuous except when the denominator vanishes. It does when $x^2 - 8x + 12 = (x - 6)(x - 2) = 0$, so f is continuous in $\mathbb{R} - \{2, 6\}$.

(ii) The function is the sum of a plynomial (continuous in $\mathbb{R}$) and the function $e^{3/x}$. The exponential is continuous everywhere and the function $3/x$ too, except for $x = 0$. Besides,

$$\lim_{x\to 0^+} e^{3/x} = \infty,$$

so f is continous in $\mathbb{R} - \{0\}$.

(iii) Polynomials are continuous in $\mathbb{R}$ and so the tangent except when its argument is an odd multiple of $\pi/2$. This means the points

$$3x + 2 = n\pi + \frac{\pi}{2} \quad \Rightarrow x = \frac{n\pi - 2}{3} + \frac{\pi}{6}, \quad n \in \mathbb{Z}.$$

f is continuous except at these infinitely many points.

(iv) Each piece of this piecwise function separately is a continuous function, so we just need to check what happens at the joints. Thus,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^3 = 0, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (|x| - x) = 0,$$

so

$$\lim_{x\to 1} f(x) = 0 = f(1).$$

And

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} (|x|-x) = 2, \qquad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} \sin(\pi x) = 0,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-1\}$.

(v) Each of the three pieces of this piecewise function is continuous (a polynomial or the absolute value of a polynomial), so we need to check just the joints. Thus,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x - 5) = 3, \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x^2 - 1| = 3,$$

so

$$\lim_{x\to 2} f(x) = 3 = f(2).$$

And

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} |x^2 - 1| = 3, \qquad \lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} x^2 = 4,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-2\}$.

(vi) The functions defining $f(x)$ for $|x| > 1$ are both polynomials — hence continuous. Within $|x| \le 1$ it is defined as $g(x) = x - \lfloor x \rfloor$. Now, $g(x) = x + 1$ for all $-1 \le x < 0$, $g(x) = x$ for all $0 \le x < 1$, and $g(1) = 0$. Thus function $f(x)$ can be redefined as

$$f(x) = \begin{cases} (x-1)^2, & x \ge 1, \\ x, & 0 \le x < 1, \\ x + 1, & x < 0. \end{cases}$$

All three pieces are continuous (polynomials), so we must look at the joints. So,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1,$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0, \qquad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1.$$

Therefore the $f(x)$ is continuous in $\mathbb{R} - \{0,1\}$.

Exercise 1.13

(i) Denoting $f(x) = x^2 - 18x + 2$, a continuous function in $\mathbb{R}$, we have $f(-1) = 21$, $f(1) = -15$, so Bolzano's theorem guarantees at least one zero in $[-1, 1]$.

(ii) Denoting $f(x) = x - \sin x - 1$, a continuous function in $\mathbb{R}$, we have $f(0) = -1$ and $f(\pi) = \pi - 1 > 0$, so Bolzano's theorem guarantees at least one zero in $[0, \pi]$.

(iii) Since $e^x > 0$, we know that $e^x + 1 > 0$, so the equation cannot have any solution in $\mathbb{R}$.

(iv) Since $-1 \le \cos x \le 1$ for all $x \in \mathbb{R}$, the equation $\cos x = -2$ cannot have any solution in $\mathbb{R}$.

A.2 Derivatives

Exercise 2.1

(i)
$$h'(x) = \frac{f(x)f'(x) + g(x)g'(x)}{\sqrt{f(x)^2 + g(x)^2}}.$$

(ii)
$$h'(x) = \frac{1}{1 + \left(\frac{f(x)}{g(x)}\right)^2} \cdot \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{f(x)^2 + g(x)^2}.$$

(iii)
$$h'(x) = f'\big(g(x)\big)g'(x)e^{f(x)} + f\big(g(x)\big)f'(x)e^{f(x)} = \left[f'\big(g(x)\big)g'(x) + f\big(g(x)\big)f'(x)\right] e^{f(x)}.$$

(iv) First of all $h(x) = \log\big(g(x)\big) + \log\big(\sin f(x)\big)$, so
$$h'(x) = \frac{g'(x)}{g(x)} + \frac{f'(x)\cos f(x)}{\sin f(x)} = \frac{g'(x)}{g(x)} + f'(x)\cot f(x).$$

(v) We first write $f(x)^{g(x)} = \exp\big\{g(x)\log f(x)\big\}$. Then
$$\begin{aligned} h'(x) &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right] \exp\big\{g(x)\log f(x)\big\} \\ &= \left[g'(x)\log f(x) + \frac{g(x)f'(x)}{f(x)}\right] f(x)^{g(x)} \\ &= f(x)^{g(x)}g'(x)\log f(x) + g(x)f'(x)f(x)^{g(x)-1}. \end{aligned}$$

(vi)
$$h'(x) = -\frac{1}{\left[\log\big(f(x) + g(x)^2\big)\right]^2} \cdot \frac{f'(x) + 2g(x)g'(x)}{f(x) + g(x)^2}.$$

Exercise 2.2

(i) $f'(x) = -\dfrac{c}{x^2}$, therefore
$$xf' + f = -\frac{c}{x} + \frac{c}{x} = 0.$$

(ii) $f'(x) = \tan x + x(1 + \tan^2 x)$, therefore
$$xf' - f - f^2 = x\tan x + x^2 - x^2\tan^2 x - x\tan x - x^2\tan^2 x = x^2.$$

(iii) $f'(x) = 3c_1\cos 3x - 3c_2\sin 3x$ and $f''(x) = -9c_1\sin 3x - 9c_2\cos 3x$, therefore
$$f'' + 9f = -9c_1\sin 3x - 9c_2\cos 3x + 9(c_1\sin 3x + c_2\cos 3x) = 0.$$

(iv) $f'(x) = 3c_1e^{3x} - 3c_2e^{-3x}$ and $f''(x) = 9c_1e^{3x} + 9c_2e^{-3x}$, therefore
$$f'' - 9f = 9c_1e^{3x} + 9c_2e^{-3x} - 9(c_1e^{3x} + c_2e^{-3x}) = 0.$$

(v) $f'(x) = 2c_1e^{2x} + 5c_2e^{5x}$ and $f''(x) = 4c_1e^{2x} + 25c_2e^{5x}$, therefore

$$\begin{aligned} f'' - 7f' + 10f &= 4c_1e^{2x} + 25c_2e^{5x} - 7(2c_1e^{2x} + 5c_2e^{5x}) + 10(c_1e^{2x} + c_2e^{5x}) \\ &= (4 - 14 + 10)e^{2x} + (25 - 35 + 10)e^{5x} = 0. \end{aligned}$$

(vi) $f'(x) = \dfrac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}$ and

$$f''(x) = \frac{(c_1e^x + e^{-x})^2 - (c_1e^x - e^{-x})^2}{(c_1e^x + e^{-x})^2} = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2,$$

therefore

$$f'' - (f')^2 = 1 - \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 + \left(\frac{c_1e^x - e^{-x}}{c_1e^x + e^{-x}}\right)^2 = 0.$$

Exercise 2.3

(i) Differentiating $f(x) = \arctan x + \arctan \dfrac{1}{x}$,

$$\frac{1}{1+x^2} + \frac{1}{1+\dfrac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0.$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x > 0$, say $x = 1$. Then $f(1) = c = \arctan 1 + \arctan 1 = 2\pi/4 = \pi/2$.

(ii) Differentiating $f(x) = \arctan \dfrac{1+x}{1-x} - \arctan x$,

$$\begin{aligned} f'(x) &= \frac{1}{1+\left(\dfrac{1+x}{1-x}\right)^2}\frac{1-x+1+x}{(1-x)^2} - \frac{1}{1+x^2} = \frac{2}{(1-x)^2+(1+x)^2} - \frac{1}{1+x^2} \\ &= \frac{2}{1-2x+x^2+1+2x+x^2} - \frac{1}{1+x^2} = \frac{2}{2+2x^2} - \frac{1}{1+x^2} = 0. \end{aligned}$$

Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x < 1$, say $x = 0$. Then $f(0) = c = \arctan 1 + \arctan 0 = \pi/4$.

(iii) Differentiating $f(x) = 2\arctan x + \arcsin \dfrac{2x}{1+x^2}$,

$$\begin{aligned} f'(x) &= \frac{2}{1+x^2} + \frac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}}\frac{2(1+x^2) - 2x\cdot 2x}{(1+x^2)^2} \\ &= \frac{2}{1+x^2} + \frac{1+x^2}{\sqrt{(1+x^2)^2-4x^2}}\frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)\sqrt{(1-x^2)^2}} \\ &\underset{(*)}{=} \frac{2}{1+x^2} + \frac{2(1-x^2)}{(1+x^2)(x^2-1)} = \frac{2}{1+x^2} - \frac{2}{1+x^2} = 0, \end{aligned}$$

where in (*) we have used the fact that $x \geq 1$ implies that $\sqrt{(1-x^2)^2} = x^2 - 1 \geq 0$. Therefore $f(x) = c$, a constant. To find out which constant we must evaluate $f(x)$ at any point $x \geq 1$, say $x = 1$. Then $f(1) = c = 2\arctan 1 + \arcsin 1 = 2\pi/4 + \pi/2 = \pi$.

Exercise 2.4 If we calculate $f'(x) = 1 + \frac{1}{3}(\sin x)^{-2/3}\cos x$ we observe that this function diverges whenever $\sin x = 0$, i.e., for $x = n\pi$ with $n \in \mathbb{Z}$. Those are the points where the tangent straight line is vertical.

Exercise 2.5 Let us calculate the derivative on the left, $f'(0^-)$ and on the right, $f'(0^+)$. Since $f(0) = 0$,

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{1}{1 + e^{1/x}} = \lim_{t\to -\infty} \frac{1}{1 + e^t} = 1,$$
$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^+} \frac{1}{1 + e^{1/x}} = \lim_{t\to \infty} \frac{1}{1 + e^t} = 0.$$

So the slope of the tangent on the left is 1 —hence it forms an angle $\pi/4$ with the X axis— and that on the right is 0 —hence it is parallel to the X axis. Thus the angle between both tangents is $\pi/4$.

Exercise 2.6 The domain of this function requires that $x + 2 \geq 0$ and $-1 \leq x + 2 \leq 1$ be satisfied simultaneously. This happens for x such that $0 \leq x + 2 \leq 1$, in other words, for $x \in [-2, -1]$. Within this domain the function is continuous because so are $x + 2$, $\sqrt{x}$, and $\cos x$ —hence its inverse— in their respective domains.

About differentiability,

$$f'(x) = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \frac{\sqrt{x+2}}{\sqrt{1-(x+2)^2}} = \frac{\arccos(x+2)}{2\sqrt{x+2}} - \sqrt{\frac{x+2}{-3-4x-x^2}},$$

which diverges when $x = -2$ and is defined only if $x^2 + 4x + 3 = (x+1)(x+3) < 0$. This happens for $x \in (-3, -1)$, an interval that overlaps with the domain excluding the point $x = -1$. Thus the derivative exits only for $x \in (-2, -1)$.

Exercise 2.7 Function $f(x)$ will be differentiable if and only if $\alpha x^2 - x + 3 \geq 0$ for all $x \in \mathbb{R}$ or $\alpha x^2 - x + 3 \leq 0$ for all $x \in \mathbb{R}$. The reason is that in either of these two cases the parabola does not cross the X axis or it just touches the axis at one point (it is only if the parabola crosses the axis that its absolute value generates points with no derivative). The condition for this to happen is that the discriminant of the parabola be ≤ 0, i.e., $1 - 12\alpha \leq 0$. Thus $\alpha \geq 1/12$.

Exercise 2.8 Function $f(x)$ is even, so it is enough to make sure that it is continuous and differentiable at $x = c$. The function will be continuous at $x = c$ if

$$a + bc^2 = \frac{1}{c}.$$

On the other hand, for $x \geq 0$ the function is

$$f(x) = \begin{cases} a + bx^2, & 0 \leq x \leq c, \\ \dfrac{1}{x}, & x > c, \end{cases}$$

so its derivative will be

$$f'(x) = \begin{cases} 2bx, & 0 \leq x < c, \\ -\dfrac{1}{x^2}, & x > c, \end{cases}$$

and therefore $f(x)$ will be differentiable at $x = c$ if

$$2bc = -\frac{1}{c^2} \qquad \Leftrightarrow \qquad b = -\frac{1}{2c^3}.$$

And from the previous equation we obtain

$$a = \frac{1}{c} - bc^2 = \frac{1}{c} + \frac{1}{2c} = \frac{3}{2c}.$$

So for $|x| < c$ the function is defined as

$$f(x) = \frac{1}{2c}\left(3 - \frac{x^2}{c^2}\right).$$

Exercise 2.9 The two pieces defining this function are continuous and differentiable within their respective sets, so the only critical point is $x = 1$. Let us first check the continuity at this point. So

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = 1, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{3-x^2}{2} = 1,$$

hence

$$\lim_{x\to 1} f(x) = 1 = f(1),$$

which proves that the function is continuous also at this point. As for differentiability,

$$\begin{aligned}
f'(1^+) &= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1^+} \frac{1-x}{x(x-1)} = -1, \\
f'(1^-) &= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{\frac{3-x^2}{2}-1}{x-1} = \lim_{x\to 1^-} \frac{1-x^2}{2(x-1)} = \lim_{x\to 1^-} \frac{(1-x)(1+x)}{2(x-1)} \\
&= \lim_{x\to 1^-} \frac{-(1+x)}{2} = -1,
\end{aligned}$$

so f is differentiable at this point and $f'(1) = -1$. Summarising, f is continuous and differentiable in $\mathbb{R}$.

A.3 Taylor Expansions

Exercise 3.1

(i) There are two ways to solve these exercises. The first one amounts to applying Taylor's formula for $P_{n,a}(x)$. For the case of $f(x) = e^x \sin x$ we have

$$\begin{aligned}
f(x) &= e^x \sin x, & f(0) &= 0,\\
f'(x) &= e^x(\sin x + \cos x), & f'(0) &= 1,\\
f''(x) &= 2e^x \cos x, & f''(0) &= 2,\\
f'''(x) &= 2e^x(\cos x - \sin x), & f'''(0) &= 2,\\
f^{(4)}(x) &= -4e^x \sin x, & f^{(4)}(0) &= 0,\\
f^{(5)}(x) &= -4e^x(\sin x + \cos x), & f^{(5)}(0) &= -4,
\end{aligned}$$

thus

$$P_{5,0}(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}.$$

The alternative way —the one we will follow here— amounts to relying upon known Taylor expansions and operate with them. For instance in this case we know that when $x \to 0$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+o(x^5), \qquad \sin x = x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5),$$

therefore, multiplying the two expressions —and collecting any power higher than x^5 as $o(x^5)$— we obtain

$$\begin{aligned}
e^x \sin x &= \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5)\right]\left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right]\\
&= \left[x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right] + \left[x^2 - \cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^3}{2} - \frac{x^5}{12} + o(x^5)\right]\\
&\quad + \left[\cancel{\frac{x^4}{6}} + o(x^5)\right] + \left[\frac{x^5}{24} + o(x^5)\right]\\
&= x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{120} + \frac{1}{24} - \frac{1}{12}\right)x^5 + o(x^5)\\
&= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + o(x^5),
\end{aligned}$$

and we get to the same result.

(ii) Now

$$e^{-x^2} = 1-x^2+\frac{x^4}{2}+o(x^5), \qquad \cos 2x = 1-\frac{(2x)^2}{2}+\frac{(2x)^4}{24}+o(x^5) = 1-2x^2+\frac{2}{3}x^4+o(x^5),$$

so multiplying and collecting equal powers,

$$\begin{aligned}
e^{-x^2}\cos 2x &= \left[1 - x^2 + \frac{x^4}{2} + o(x^5)\right]\left[1 - 2x^2 + \frac{2}{3}x^4 + o(x^5)\right]\\
&=1 - (1+2)x^2 + \left(\frac{1}{2} + 2 + \frac{2}{3}\right)x^4 + o(x^5)\\
&=1 - 3x^2 + \frac{19}{6}x^4 + o(x^5).
\end{aligned}$$

Thus
$$P_{5,0}(x) = 1 - 3x^2 + \frac{19}{6}x^4.$$

(iii) Using the trigonometric identity
$$\sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta+\phi) + \sin(\theta-\phi)\right]$$
we can write
$$\sin x \cos 2x = \frac{1}{2}\left(\sin 3x - \sin x\right).$$
Now, since for $z \longrightarrow 0$
$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + o(z^5),$$
then
$$\sin x\cos 2x = \frac{1}{2}\left(3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 - x + \frac{x^3}{6} - \frac{x^5}{120}\right) + o(x^5)$$

(iv) In this case
$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+o(x^5), \qquad \log(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+o(x^5),$$
so
$$\begin{aligned}
e^x\log(1-x) &= -x\left[1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+o(x^4)\right]\left[1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+o(x^4)\right]\\
&= -x\left[1+\left(1+\frac{1}{2}\right)x+\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{3}\right)x^2+\left(\frac{1}{6}+\frac{1}{4}+\frac{1}{3}+\frac{1}{4}\right)x^3\right.\\
&\quad\left.+\left(\frac{1}{24}+\frac{1}{12}+\frac{1}{6}+\frac{1}{4}+\frac{1}{5}\right)x^4+o(x^4)\right]\\
&= -x-\frac{3}{2}x^2-\frac{4}{3}x^3-x^4-\frac{89}{120}x^5+o(x^5).
\end{aligned}$$
Therefore
$$P_{5,0}(x) = -x-\frac{3}{2}x^2-\frac{4}{3}x^3-x^4-\frac{89}{120}x^5.$$

(v) Since $\sin^2 x = (1-\cos 2x)/2$,
$$\sin^2 x = \frac{1}{2}\left[\cancel{1}-\cancel{1}+\frac{(2x)^2}{2}-\frac{(2x)^4}{24}+o(x^5)\right] = x^2-\frac{x^4}{3}+o(x^5),$$
hence
$$P_{5,0}(x) = x^2 - \frac{x^4}{3}.$$

(vi) We know that
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots,$$
therefore
$$\frac{1}{1-x^3} = 1+x^3+o(x^5),$$

which implies $P_{5,0}(x) = 1 + x^3$.

Exercise 3.2 The Taylor polynomial $P_{4,4}(x)$ of $P(x) = x^4 - 5x^3 + x^2 - 3x + 4$ is obtained through

$$\begin{aligned}
P(x) &= x^4 - 5x^3 + x^2 - 3x + 4, & P(4) &= -56,\\
P'(x) &= 4x^3 - 15x^2 + 2x - 3, & P'(4) &= 21,\\
P''(x) &= 12x^2 - 30x + 2, & P''(4) &= 74,\\
P'''(x) &= 24x - 30, & P'''(4) &= 66,\\
P^{(4)}(x) &= 24, & P^{(4)}(4) &= 24.
\end{aligned}$$

Hence

$$P(x) = -56 + 21(x-4) + 37(x-4)^2 + 11(x-4)^3 + (x-4)^4.$$

Exercise 3.3

(i) The polynomial must be expressed in powers of $t = x + 1$, so if we write

$$\frac{1}{x} = \frac{1}{t-1} = -\frac{1}{1-t} = -1 - t - t^2 - \cdots - t^n + \cdots$$

we immediately obtain $P_{n,-1}(x) = -1-(x+1)-(x+1)^2-\cdots-(x+1)^n$.

(ii) Since

$$\begin{aligned}
e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2} + \cdots + \frac{(-2x)^{n-1}}{(n-1)!} + o(x^{n-1})\\
&= 1 - 2x + 2x^2 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^{n-1} + o(x^{n-1})
\end{aligned}$$

then

$$xe^{-2x} = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n + o(x^n).$$

Thus

$$P_{n,0}(x) = x - 2x^2 + 2x^3 + \cdots + (-1)^{n-1}\frac{2^{n-1}}{(n-1)!}x^n.$$

(iii) We can expand $(1 + e^x)^2 = 1 + 2e^x + e^{2x}$, so

$$\begin{aligned}
(1+e^x)^2 =& 1 + 2\left[1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + o(x^n)\right] + \left[1 + 2x + \frac{(2x)^2}{2} + \cdots + \frac{(2x)^n}{n!} + o(x^n)\right]\\
=& 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n + o(x^n),
\end{aligned}$$

from which

$$P_{n,0}(x) = 4 + 4x + 3x^2 + \cdots + \frac{2+2^n}{n!}x^n.$$

(iv) We must express the polynomial in powers of $t = x - \pi$, therefore

$\sin x = \sin(\pi + t) = -\sin t$, and

$$\sin x = -t + \frac{t^3}{6} - \frac{t^5}{120} + \cdots + (-1)^n \frac{t^{2n-1}}{(2n-1)!} + o(t^{2n-1}).$$

Thus

$$P_{2n,\pi}(x) = P_{2n-1,\pi}(x) = -(x-\pi) + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)^5}{120} + \cdots + (-1)^n \frac{(x-\pi)^{2n-1}}{(2n-1)!}.$$

Exercise 3.4 Since $\sin x = x + o(x)$, then $f(x) = 1 + x^4 + o(x^4)$, when $x \to 0$. Thus $P_{4,0}(x) = 1 + x^4$. Accordingly f has a local minimum at $x = 0$.

Exercise 3.5

(i) Let us consider the function

$$f(x) = \frac{1}{\sqrt{1+x}}.$$

The value we want to obtain is $f(0.1)$. The Taylor expansion for this function near $a = 0$ follows from

$$\begin{aligned}
f(x) &= (1+x)^{-1/2}, & f(0) &= 1,\\
f'(x) &= -\frac{1}{2}(1+x)^{-3/2}, & f'(0) &= -\frac{1}{2},\\
f''(x) &= \frac{3}{4}(1+x)^{-5/2}, & f''(0) &= \frac{3}{4},\\
f'''(x) &= -\frac{15}{8}(1+x)^{-7/2}, & f'''(0) &= -\frac{15}{8},\\
f^{(4)}(x) &= \frac{105}{16}(1+x)^{-9/2},
\end{aligned}$$

which implies

$$P_{3,0}(x) = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3, \qquad R_{3,0}(x) = \frac{35}{128}\left(\frac{1}{\sqrt{1+\theta x}}\right)^9 x^4, \quad 0 < \theta < 1.$$

Now $P_{3,0}(0.1) = 0.9534375$ and since $\sqrt{1+\theta x} > 1$ for every $x > 0$,

$$|R_{3,0}(x)| < \frac{35}{128}x^4 \quad \Rightarrow \quad |R_{3,0}(0.1)| < 2.7 \times 10^{-5}.$$

Hence $1/\sqrt{1.1} = 0.9534(3)$ —where the figure in brackets may be affected by the error. (The exact value is $1/\sqrt{1.1} = 0.953462589\ldots$)

(ii) Consider the function $f(x) = \sqrt[3]{27+x}$ =. Then $\sqrt[3]{28} = f(1)$. To ontain the second degree Taylor expansion around $a = 0$ we calculate

$$\begin{aligned}
f(x) &= (27+x)^{1/3}, & f(0) &= 3,\\
f'(x) &= \frac{1}{3}(27+x)^{-2/3}, & f'(0) &= \frac{1}{27},\\
f''(x) &= -\frac{2}{9}(27+x)^{-5/3}, & f''(0) &= -\frac{2}{2187},\\
f'''(x) &= \frac{10}{27}(27+x)^{-8/3},
\end{aligned}$$

from which

$$P_{2,0}(x) = 3+\frac{x}{27}-\frac{x^2}{2187}, \qquad R_{2,0}(x) = \frac{5}{81}\frac{x^3}{\left(\sqrt[3]{27+\theta x}\right)^8}, \quad 0 < \theta < 1.$$

Now $P_{2,0}(1) = 3.03657979$ and since $\sqrt[3]{27+\theta x} > \sqrt[3]{27} = 3$ for every $x > 0$,

$$|R_{2,0}(x)| < \frac{5x^3}{531441} \qquad \Rightarrow \qquad |R_{2,0}(1)| < \frac{5}{531441} = 0.9408\times 10^{-5}.$$

Hence $\sqrt[3]{28} = 3.0365(8)$. (As a matter of fact $\sqrt[3]{28} = 3.036588972\ldots$)

(iii) Taking now $f(x) = \log(1+x)$ the fourth-degree Taylor polynomial is

$$P_{4,0} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

For $x = 1/2$ we have

$$P_{4,0}(1/2) = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} = 0.4010416666666\ldots$$

Now the fifth derivative is $f^{(v)}(x) = 24(1+x)^{-5}$ which has a maximum value of 24 in $(0, 1/2)$ at $x = 0$. Hence,

$$|R_{4,0}(x)| < \frac{24}{5!}x^5 = \frac{1}{5}x^5,$$

and

$$|R_{4,0}(1/2)| < \frac{1}{5}\frac{1}{32} = 0.00625.$$

Therefore we can write $\log(3/2) = 0.40(1)$. And in fact the true value is $0.4054651081081644\ldots$.

Exercise 3.6

(i) Since for $x \to 0$

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3),$$

then

$$P_{3,0}(x) = 2 + x + \frac{x^3}{6}.$$

(ii) First of all $(\cos x)^{(4)} = \cos x$ and $(e^x)^{(4)} = e^x$, so $f^{(4)}(x) = f(x)$. Therefore

$$R_{3,0}(x) = \frac{\cos\theta x + e^{\theta x}}{24}x^4, \quad 0 < \theta < 1.$$

Now $|\cos\theta x| \le 1$ and $e^{\theta x} \le \max\{e^x, 1\}$. Thus for $-1/4 \le x \le 1/4$

$$|R_{3,0}(x)| < \frac{1+e^{1/4}}{24}\left(\frac{1}{4}\right)^4 = 3.72\times 10^{-4}.$$

Exercise 3.7 The reminder of the Taylor expansion of $f(x) = e^x$ around $a = 0$ is

$$R_{n,0}(x) = \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1,$$

so an upper bound for $-1 \leq x \leq 1$ will be

$$|R_{n,0}(x)| < \frac{e}{(n+1)!}.$$

If we want to have three exact decimal places the error should be smaller than 10^{-3}, so we must look for the smallest n for which $(n+1)! > 10^3$. Since $6! = 720$ and $7! = 5040$ then $n = 6$.

Exercise 3.8

(i) To begin with

$$f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

and since

$$\cos t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \quad t \in \mathbb{R},$$

substituting we obtain

$$\begin{aligned} f(x) &= \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[-\sum_{n=1}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n} \frac{x^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n-1} \frac{x^{2n}}{(2n)!}, \quad x \in \mathbb{R}. \end{aligned}$$

(ii) We can rewrite

$$f(x) = \log \sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)$$

and use

$$\log(1+t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^n}{n}, \quad |t| < 1,$$

to obtain

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1}+1}{2}\right] \frac{x^n}{n}, \quad |x| < 1.$$

But

$$\frac{(-1)^{n+1}+1}{2} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

Therefore

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

(iii) We can rewrite

$$f(x) = \frac{x}{a} \cdot \frac{1}{1 + bx/a}.$$

Now since

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad |t| < 1,$$

then

$$f(x) = \frac{x}{a}\sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^n}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{b^n}{a^{n+1}}x^{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{b^{n-1}}{a^n}x^n, \quad |x| < \left|\frac{a}{b}\right|.$$

(iv) We can express

$$f(x) = \frac{1}{2}\frac{1}{1 - x^2/2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{x^{2n}}{2^{n+1}},$$

and the converge requires $x^2/2 < 1$, i.e., $|x| < \sqrt{2}$.

A.4 Local Behavior of Functions

Exercise 4.1

(a) f is continuous in $\mathbb{R}$ because so are polynomials and the absolute value function. As for differentiability, we can express f in a piecewise description as

$$f(x) = \begin{cases} 4x^3 - x^4 - 1, & 0 < x < 4, \\ x^4 - 4x^3 - 1, & \text{otherwise,} \end{cases}$$

separating out the cases where $x^3(x-4) < 0$ from those where $x^3(x-4) \geq 0$. Both pieces are differentiable (they are polynomials), so we must check the joints. Since

$$f'(x) = \begin{cases} 12x^2 - 4x^3, & 0 < x < 4, \\ 4x^3 - 12x^2, & x < 0 \text{ or } x > 4, \end{cases}$$

we have $f'(0-) = f'(0^+) = 0$, so f is differentiable at $x = 0$, but $f'(4^-) = -64$, and $f'(4^+) = 64$, so f is not differentiable at $x = 4$. Summarising, f is continuous in $\mathbb{R}$ and differentiable in $\mathbb{R} - \{4\}$.

(b) First of all we need to look where $f'(x) = 0$. This means

$$4x^2(3 - x) = 0 \quad \Rightarrow \quad x = 0,\ x = 3.$$

If $x < 0$ but close to $x = 0$ then $f'(x) = 4x^2(x-3) < 0$; if $x > 0$ but close to $x = 0$ then $f'(x) = 4x^2(3-x) > 0$. Therefore f has a local minimum at $x = 0$. On the other hand, if $x < 3$ then $f'(x) = 4x^2(3-x) > 0$ and if $x > 3$ then $f'(x) = 4x^2(3-x) < 0$, so f has a local maximum at $x = 3$.
But this is not the whole story because f is not differentiable at $x = 4$ —hence $x = 4$ cannot be a solution to $f'(x) = 0$. We need to check this point separately. Now, $f(4) = -1$, but for any $x \neq 4$ near $x = 4$ we have $f(x) = |x^3(x-4)| - 1 > -1$, so $x = 4$ is a local minimum.
Finally, -1 is the smallest value that $f(x)$ can take, and $f(0) = f(4) = -1$, so both, at $x = 0$ and at $x = 4$, function $f(x)$ reaches its absolute minimum. There is no absolute maximum though, because the function grows indefinitely as $x \longrightarrow \pm\infty$.

(c) $f(0) = -1$ and $f(1) = 2$, so Bolzano's theorem guarantees that there is at least one solution to $f(x) = 0$ in $(0, 1)$. On the other hand, in $(0, 1)$ we have $f'(x) = 4x^2(3-x) > 0$ so the function is monotonically increasing. Therefore the solution is unique.

Exercise 4.2

(a) The amount of material is proportional to the surface of the can, which is given by the formula $S = 2\pi r^2 + 2\pi r h$. But cans have all the same volume $V = \pi r^2 h$, so $h = V/\pi r^2$ and thefore

$$S = 2\pi\left(r^2 + \frac{V}{\pi r}\right).$$

Minimising the surface amounts to minimising the function

$$f(r) = r^2 + \frac{V}{\pi r}.$$

This is a differentiable function for all $r > 0$, so the minimum is reached at a solution of

$$f'(r) = 2r - \frac{V}{\pi r^2} = 0 \quad \Rightarrow \quad r^3 = \frac{V}{2\pi} \quad \Rightarrow \quad r = \left(\frac{V}{2\pi}\right)^{1/3}$$

and

$$h = \frac{V}{\pi r^2} = \left(\frac{4V}{\pi}\right)^{1/3}.$$

(b) Lead is proportional to the surface. If the side of the square base is a and the height h, then the surface will be $S = a^2 + 4ah$. The volume constraint, $32 = a^2 h$, implies $h = 32/a^2$, so

$$S = a^2 + \frac{128}{a} = f(a).$$

Now,

$$f'(a) = 2a - \frac{128}{a^2} \quad \Rightarrow \quad a^3 = 64 \quad \Rightarrow \quad a = 4, \quad h = 2.$$

(c) We can eliminate $y = 20 - x$, so the function to maximise is

$$f(x) = x^2(20 - x)^3.$$

Now,

$$f'(x) = 2x(20-x)^3 - 3x^2(20-x)^2 = x(20-x)^2(40-2x-3x) = 5x(20-x)^2(8-x) = 0.$$

The two solutions $x = 0$, $x = 20$ clearly minimise the function. The maximum is then $x = 8$ and $y = 12$.

(d) If x is half the horizontal side of the rectangle, then

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

is half the vertical side. Then the area of the rectangle is

$$A = 4xy = 4bx\sqrt{1 - \frac{x^2}{a^2}}.$$

Maximising this area is tantamount to maximising

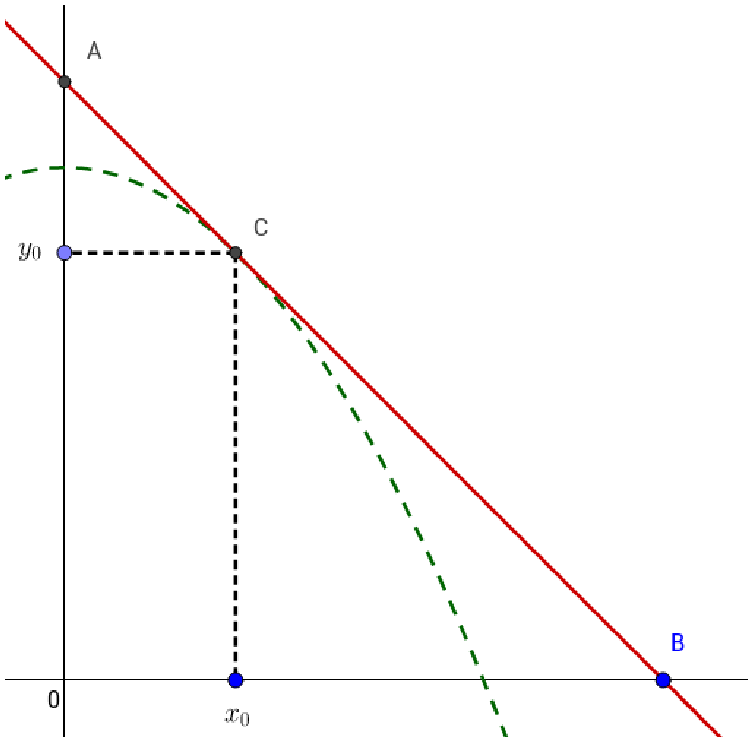
$$f(x) = \frac{A^2}{16b^2} = x^2 - \frac{x^4}{a^2},$$

which means solving the equation

$$f'(x) = 2x - \frac{4x^3}{a^2} = 2x\left(1 - \frac{2x^2}{a^2}\right) = 0.$$

One solution is $x = 0$ —which is obviously not the right one— and the other two solutions are $x = \pm a/\sqrt{2}$. Clearly the one that maximises the area has to be $x = a/\sqrt{2}$.

(e) The picture illustrates how to construct the described triangle:



We can select an arbitrary point on the parabola, $(x_0, 6 - x_0^2)$. The slope of the tangent at that point will be $m = -2x_0$ (obtained differentiating $6 - x^2$), so the equation of the tangent straight line will be

$$y = 6 - x_0^2 - 2x_0(x - x_0) = 6 + x_0^2 - 2x_0 x.$$

Now, this straight line meets the Y axis at $A(0, 6 + x_0^2)$, and the X axis at $B\big((6 + x_0^2)/2x_0, 0\big)$, so the area of the triangle will be

$$A = \frac{(6 + x_0^2)^2}{4x_0} = \frac{9}{x_0} + 3x_0 + \frac{x_0^3}{4} = f(x_0).$$

Minimising the area means solving

$$f'(x_0) = -\frac{9}{x_0^2} + 3 + \frac{3x_0^2}{4} = \frac{3(x_0^4 + 4x_0^2 - 12)}{4x_0^2} = \frac{3(x_0^2 + 6)(x_0^2 - 2)}{4x_0^2} = 0.$$

The only meaningful solution to this equation is $x_0 = \sqrt{2}$.

(f) The area of the triangle at the base is $a^2\sqrt{3}/4$, and that of the lateral rectangles $3ah$, so the total cost will be

$$C = 0.20 \times a^2 \frac{\sqrt{3}}{4} + 0.10 \times 3ah = 0.10 \times \sqrt{3}\left(\frac{a^2}{2} + \sqrt{3}ah\right).$$

Since $128 = ha^2\sqrt{3}/4$ we get $\sqrt{3}ah = 512/a$, so $C = 0.10 \times \sqrt{3} f(a)$, where

$$f(a) = \frac{a^2}{2} + \frac{512}{a}.$$

The value of a minimising cost will be a solution of

$$f'(a) = a - \frac{512}{a^2} = 0 \quad \Rightarrow \quad a^3 = 512 \quad \Rightarrow \quad a = 8.$$

(g) For a given $0 \leq x \leq 2$ the corresponding y on the circumference is given by

$$y = \sqrt{1-(x-1)^2} = \sqrt{x(2-x)}.$$

Thus, the three points of the triangle are $A(0,0)$, $B\left(x, \sqrt{x(2-x)}\right)$, $C(x,0)$. The area of the triangle will then be $S = x\sqrt{x(2-x)}/2 = x^{3/2}(2-x)^{1/2}/2$. So maximising this area is tantamount to maximising

$$f(x) = 4S^2 = x^3(2-x) = 2x^3 - x^4.$$

The corresponding x will be a solution of

$$f'(x) = 6x^2 - 4x^3 = 2x^2(3-2x) = 0.$$

The only meaningful solution is $x = 3/2$.

(h) Triangle similarity implies

$$\frac{y_0+\beta}{x_0+\alpha} = \frac{\beta}{x_0} \quad \Rightarrow \quad x_0y_0 + \cancel{\beta x_0} = \cancel{\beta x_0} + \beta\alpha \quad \Rightarrow \quad \beta = \frac{x_0y_0}{\alpha}.$$

(i) The length of segment AB is

$$\ell = \sqrt{(x_0+\alpha)^2 + (y_0+\beta)^2} = \sqrt{(x_0+\alpha)^2 + \left(y_0 + \frac{x_0y_0}{\alpha}\right)^2} = \sqrt{(x_0+\alpha)^2 + \frac{y_0^2}{\alpha^2}(x_0+\alpha)^2}$$
$$= (x_0+\alpha)\sqrt{1+\frac{y_0^2}{\alpha^2}}.$$

So minimising ℓ is tantamount to minimising

$$f(\alpha) = \ell^2 = (x_0+\alpha)^2\left(1+\frac{y_0^2}{\alpha^2}\right).$$

Differentiating

$$f'(\alpha) = 2(x_0+\alpha)\left(1+\frac{y_0^2}{\alpha^2}\right) - 2(x_0+\alpha)^2\frac{y_0^2}{\alpha^3} = 2(x_0+\alpha)\left(1+\cancel{\frac{y_0^2}{\alpha^2}} - \frac{x_0y_0^2}{\alpha^3} - \cancel{\frac{y_0^2}{\alpha^2}}\right)$$
$$= 2(x_0+\alpha)\left(1-\frac{x_0y_0^2}{\alpha^3}\right) = 0.$$

This equation has the solution

$$\alpha = \left(x_0y_0^2\right)^{1/3}, \qquad \beta = \frac{x_0y_0}{\alpha} = \left(x_0^2y_0\right)^{1/3}.$$

(ii) The sum of segments OA and OB is

$$f(\alpha) = x_0 + \alpha + y_0 + \beta = x_0 + y_0 + \alpha + \frac{x_0y_0}{\alpha}.$$

Differentiating

$$f'(\alpha) = 1-\frac{x_0 y_0}{\alpha^2} = 0 \quad \Rightarrow \quad \alpha = (x_0 y_0)^{1/2}, \quad \beta = \frac{x_0 y_0}{\alpha} = \left(x_0 y_0\right)^{1/2}.$$

(iii) The area of the triangle is

$$A = \frac{1}{2}(x_0+\alpha)(y_0+\beta) = \frac{1}{2}(x_0+\alpha)\left(y_0 + \frac{x_0 y_0}{\alpha}\right) = \frac{y_0}{2}\frac{(x_0+\alpha)^2}{\alpha} = \frac{y_0}{2}\left(\frac{x_0^2}{\alpha} + 2x_0 + \alpha\right).$$

Minimising the area implies minimising

$$f(\alpha) = \frac{2A}{y_0} = \frac{x_0^2}{\alpha} + 2x_0 + \alpha.$$

Differentiating

$$f'(\alpha) = -\frac{x_0^2}{\alpha^2} + 1 = 0 \quad \Rightarrow \quad \alpha = x_0, \quad \beta = \frac{x_0 y_0}{\alpha} = y_0.$$

Exercise 4.3

(a) For $a = 1$ the inequality becomes a trivial equality. For $a > 1$ take the function

$$f(x) = (1+x)^a - 1 - ax.$$

Differentiating,

$$f'(x) = a(1+x)^{a-1} - a = 0 \quad \Rightarrow \quad (1+x)^{a-1} = 1 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = a(a-1)(1+x)^{a-2} \quad \Rightarrow \quad f''(0) = a(a-1) > 0$$

we conclude that $x = 0$ is a minimum —the absolute minimum if $x > -1$—, therefore $f(x) \geq f(0) = 0$ for every $x > -1$. This means

$$(1+x)^a \geq 1 + ax.$$

(b) Take the function

$$f(x) = e^x - 1 - x.$$

Differentiating,

$$f'(x) = e^x - 1 = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —which is absolute in this case because there is no other one in $\mathbb{R}$. Therefore $f(x) \geq f(0) = 0$ for every $x \in \mathbb{R}$, i.e.,

$$e^x \geq 1 + x.$$

(c) Take the function

$$f(x) = \log(1+x) - \frac{x}{1+x}.$$

Differentiating,

$$f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = -\frac{1}{(1+x)^2} + \frac{2}{(1+x)^3} = \frac{1-x}{(1+x)^3} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —which is absolute in this case because there is no other one when $x > -1$. Therefore $f(x) \geq f(0) = 0$ for every $x > -1$. This proves the first inequality. As for the second, take

$$g(x) = x - \log(1+x)$$

and differentiate:

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} = 0 \quad \Rightarrow \quad x = 0,$$

so $x = 0$ is a local extremum. From the second derivative,

$$f''(x) = \frac{1}{(1+x)^2} \quad \Rightarrow \quad f''(0) = 1 > 0,$$

we conclude that $x = 0$ is a minimum —again absolute—, so $f(x) \geq f(0) = 0$ for every $x > -1$. This proves the second inequality.

Exercise 4.4

(i) The polynomial $f(x) = x^7 + 4x - 3 \sim x^7$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. What we need to know is to figure out how many times $f(x)$ bends up and down and from that determining the number of times it crosses the X axis. Now,

$$f'(x) = 7x^6 + 4 > 0$$

for all $x \in \mathbb{R}$, therefore $f(x)$ increases monotonically. The conclusion is that there is only *one* solution.

(ii) Similarly to the previous exercise, $f(x) = x^5 - 5x + 6 \sim x^5$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 5x^4 - 5 = 0 \quad \Rightarrow \quad x = \pm 1,$$

and from the second derivative

$$f''(x) = 20x^3 \quad \Rightarrow \quad f''(1) = 20 > 0, \quad f''(-1) = -20 < 0,$$

so we conclude that $x = -1$ is a local minimum and $x = 1$ a local maximum. But $f(1) = 2 > 0$ and $f(-1) = 10 > 0$, so the local minimum is above the X axis. In conclusion, there is only *one* solution.

(iii) $f(x) = x^4 - 4x^3 - 1 \sim x^4$ as $x \to \pm\infty$, so $f(x) \to \infty$ when $x \to \pm\infty$. It is not guaranteed that there is even a single solution. From the derivative,

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

we conclude that $x = 0$ and $x = 3$ may be extrema. $f'(x) < 0$ around $x = 0$ (at both sides), so it is an inflection point. However, close to $x = 3$ we have $f'(x) < 0$ for $x < 3$ and $f'(x) > 0$ for $x > 3$, so at $x = 3$ the polynomial reaches its absolute minimum $f(3) = -28$. Since this value is below the X axis, $f(x)$ has to cross it twice. Therefore there are *two* solutions to the equation.

(iv) The function $f(x) = 2x - 1 - \sin x \sim 2x$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Thus $f(x) = 0$ at at least one point. Now,

$$f'(x) = 2 - \cos x > 0 \quad \text{for all } x \in \mathbb{R},$$

so $f(x)$ monotonically increases. Therefore there is only *one* solution.

(v) Let us first rewite the equation. Taking logarithms the equation becomes

$$f(x) = x \log x - \log 2 = 0.$$

$f(1) = -\log 2 < 0$ and $f(x) \to \infty$ as $x \to \infty$, so $f(x)$ vanishes at one point at least. Now,

$$f'(x) = \log x + 1,$$

which is $f'(x) < 0$ for $x < 1/e$ and $f'(x) > 0$ for $x > 1/e$. In other words, $f'(x) > 0$ in the interval $[1, \infty)$, so $f(x)$ monotonically increases in that interval. Therefore there is only *one* solution.

(vi) Writing the equation

$$f(x) = x^2 + \log x = 0$$

we have $f(1) = 1 > 0$, and $f(x) \sim x^2$ as $x \to \pm\infty$, so $f(x) \to \infty$ as $x \to \pm\infty$. There is no guarantee that the equation has even a single solution in that interval. From the derivative,

$$f'(x) = 2x + \frac{1}{x} = \frac{2x^2 + 1}{x}$$

we conclude that $f'(x) > 0$ in $(1, \infty)$, so $f(x)$ increases monotonically. Therefore the equation has *no* solution in that interval.

Exercise 4.5 Since $\sin x = x + o(x)$, then $f(x) = 1 + x^4 + o(x^4)$, when $x \to 0$. Thus $P_{4,0}(x) = 1 + x^4$. Accordingly f has a local minimum at $x = 0$.

Exercise 4.6 Let us compute two derivatives of h:

$$h' = (f' \circ g)g', \qquad h'' = (f' \circ g)' g' + (f' \circ g) g'' = (f'' \circ g)(g')^2 + (f' \circ g) g''.$$

Since f is convex $f'' \circ g > 0$; since f is increasing $f' \circ g > 0$; since g is convex $g'' > 0$; and of course $(g')^2 \geq 0$. Therefore $h'' > 0$, hence h is convex.

Exercise 4.7

(i) $f(x) = x^{5/3} - 2x^{2/3}$, so

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{4}{3}x^{-1/3}, \qquad f''(x) = \frac{10}{9}x^{-1/3} + \frac{4}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}\left(x + \frac{2}{5}\right).$$

Since $x^{-4/3} > 0$ for all $x \neq 0$, then $f(x)$ is concave for $x < -2/5$ and convex in $-2/5 < x < 0$ and $x > 0$. At $x = -2/5$ it has an inflection point, and at $x = 0$ the function has a nondifferentiable cusp.

(ii) $f(x)$ is not differentiable at $x = 0$. Now, for $x > 0$

$$f(x) = xe^x, \qquad f'(x) = (x+1)e^x, \qquad f''(x) = (x+2)e^x,$$

so the funtion is always convex. On the other hand, the function is even (because $f(-x) = f(x)$), so it is convex also for $x < 0$.

(iii) $x^2 - 6x + 8 = (x-2)(x-4)$, so the domain of this function is $(-\infty, 2) \cup (4, \infty)$. On the other hand, in its domain

$$f(x) = \log(x^2 - 6x + 8) = \log|x^2 - 6x + 8| = \log|x-2| + \log|x-4|,$$

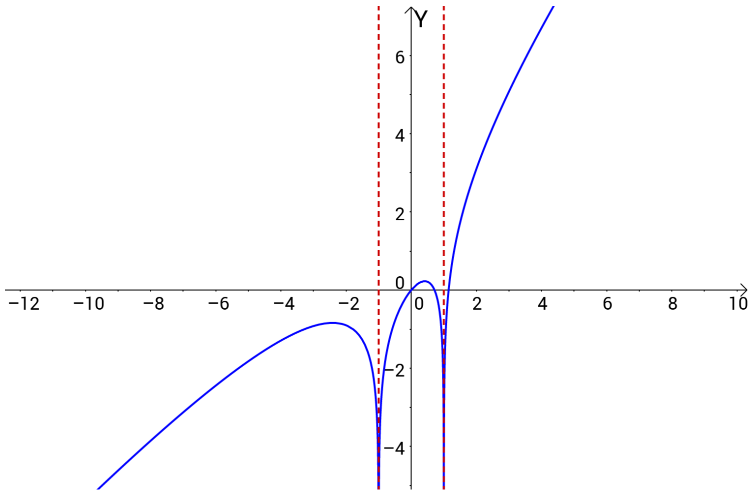
so

$$f'(x) = \frac{1}{x-2} + \frac{1}{x-4}, \qquad f''(x) = -\frac{1}{(x-2)^2} - \frac{1}{(x-4)^2},$$

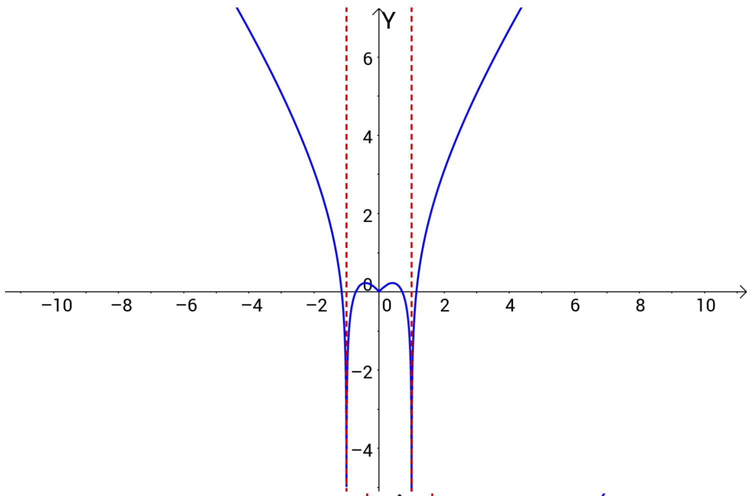
and then we have $f''(x) < 0$ in the whole domain of the function. Thus $f(x)$ is concave.

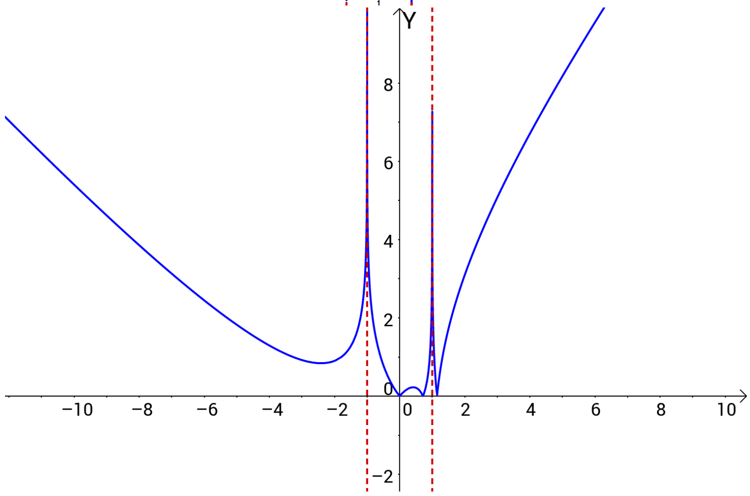
Exercise 4.8

(i) $f(x) = x + \log|x^2 - 1|$:



(ii) $g(x) = f(|x|)$ $\qquad$ $h(x) = |f(x)|$:

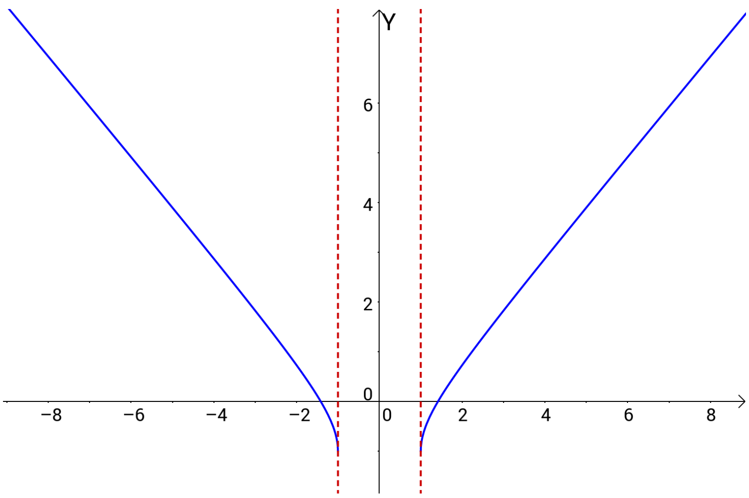




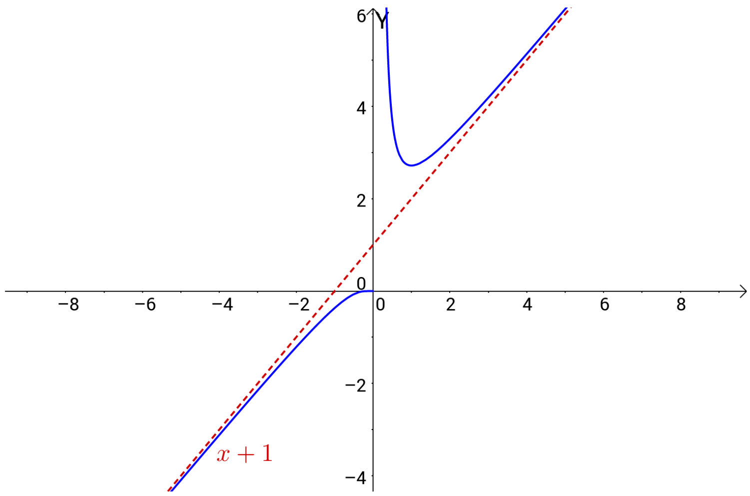
Exercise 4.9

(i) $f(x) = e^x \sin x$: this function oscillates between $y = e^x$ and $y = -e^x$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

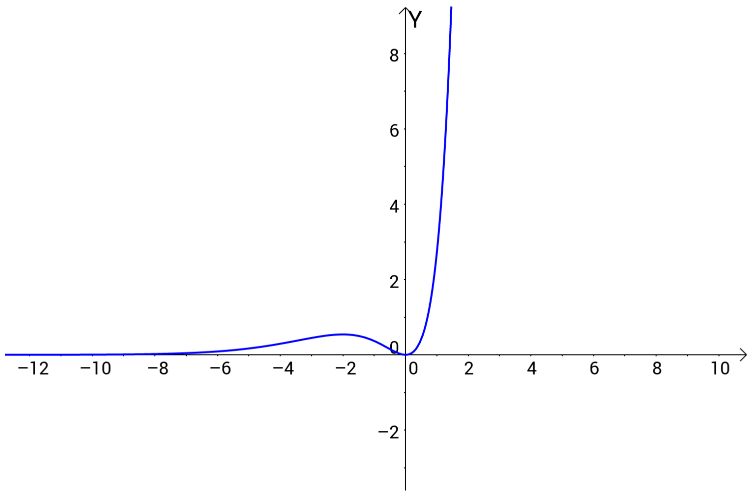
(ii) $f(x) = \sqrt{x^2 - 1} - 1$:



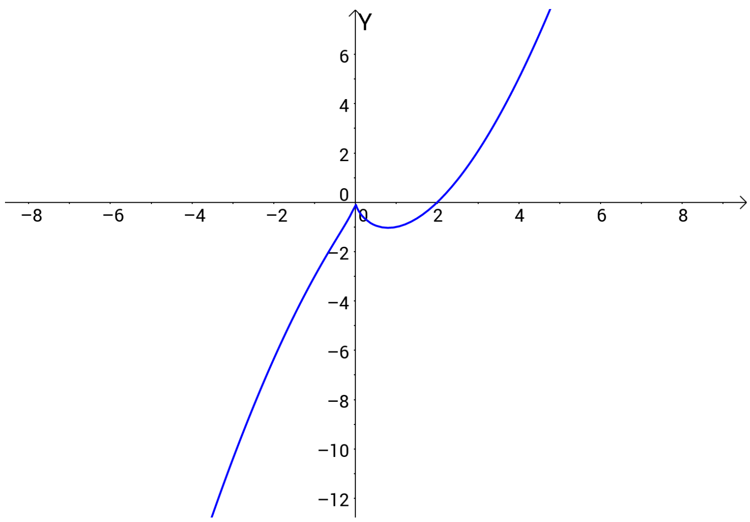
(iii) $f(x) = xe^{1/x}$:



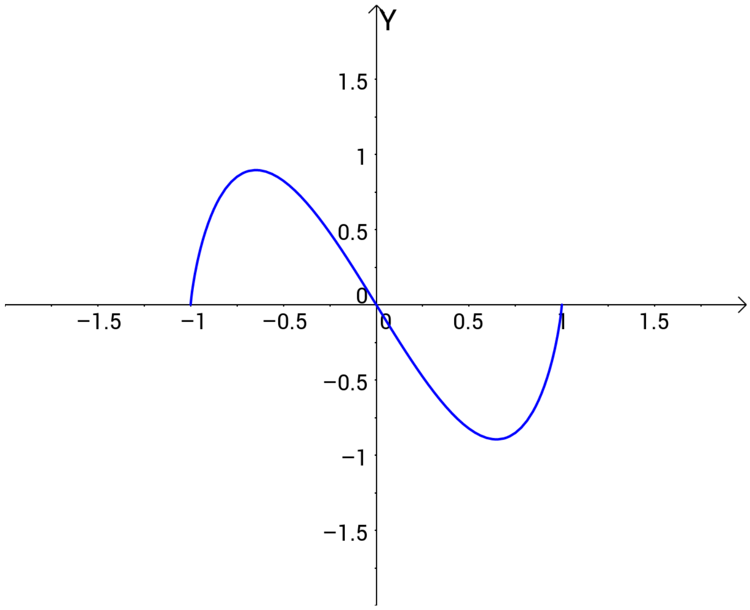
(iv) $f(x) = x^2 e^x$:



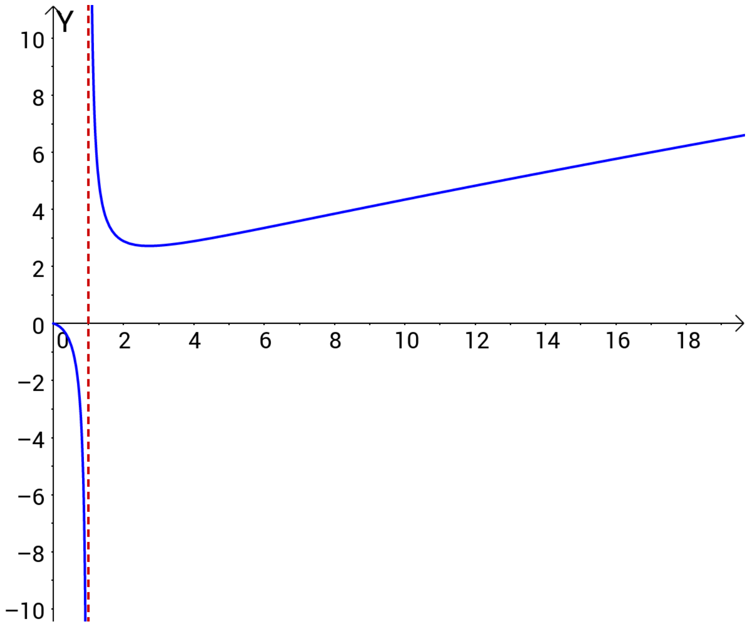
(v) $f(x) = (x-2)x^{2/3}$:



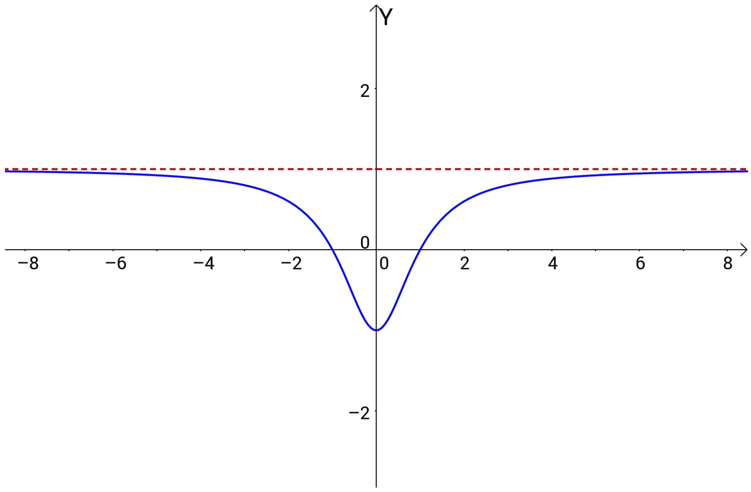
(vi) $f(x) = (x^2-1)\log\left(\dfrac{1+x}{1-x}\right)$:



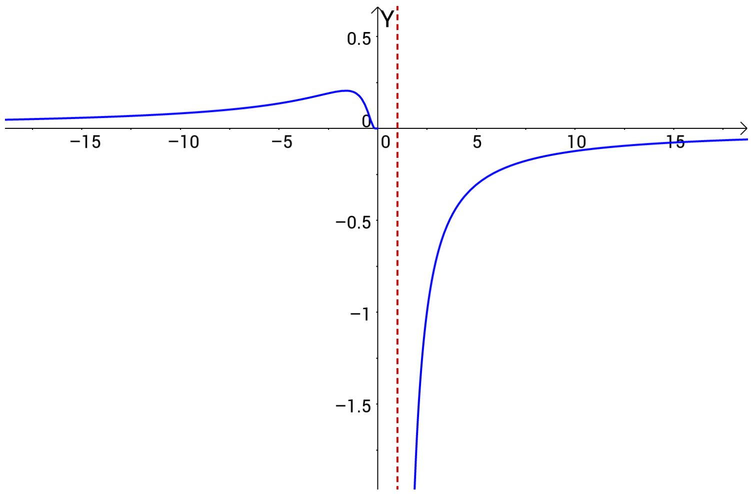
(vii) $f(x) = \dfrac{x}{\log x}$:



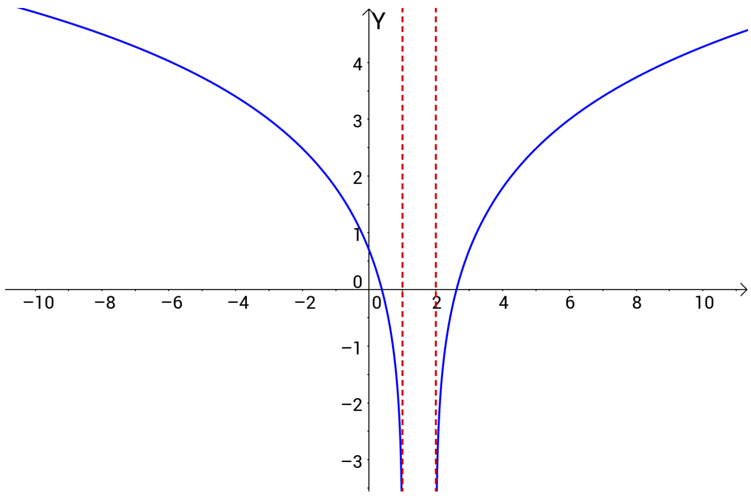
(viii) $f(x) = \dfrac{x^2-1}{x^2+1}$:



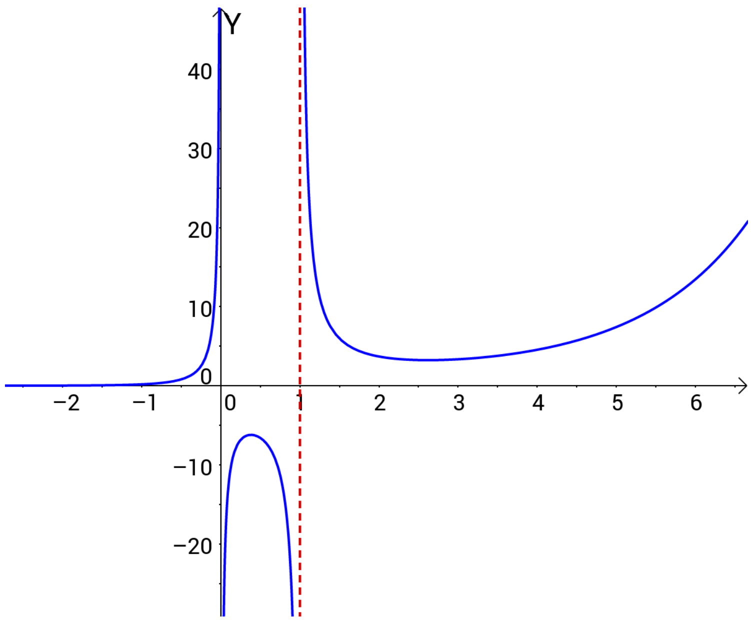
(ix) $f(x) = \dfrac{e^{1/x}}{1-x}$:



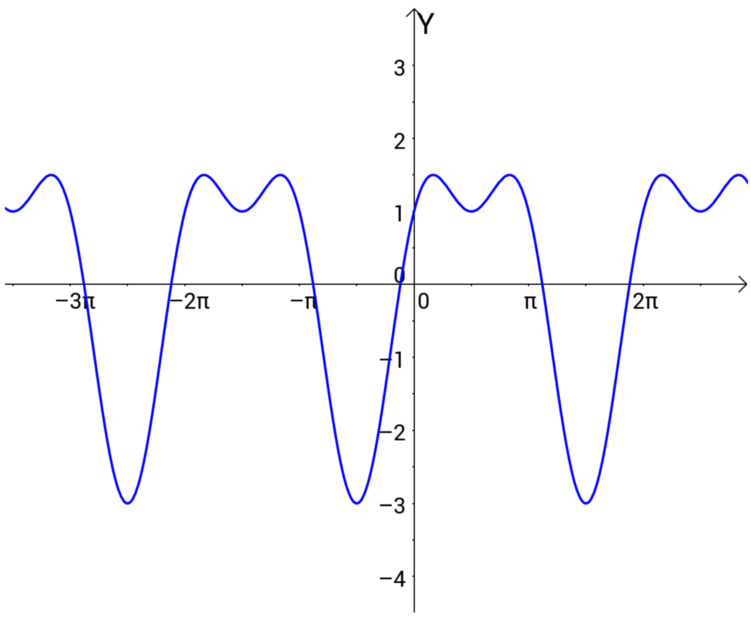
(x) $f(x) = \log\left[(x-1)(x-2)\right]$:



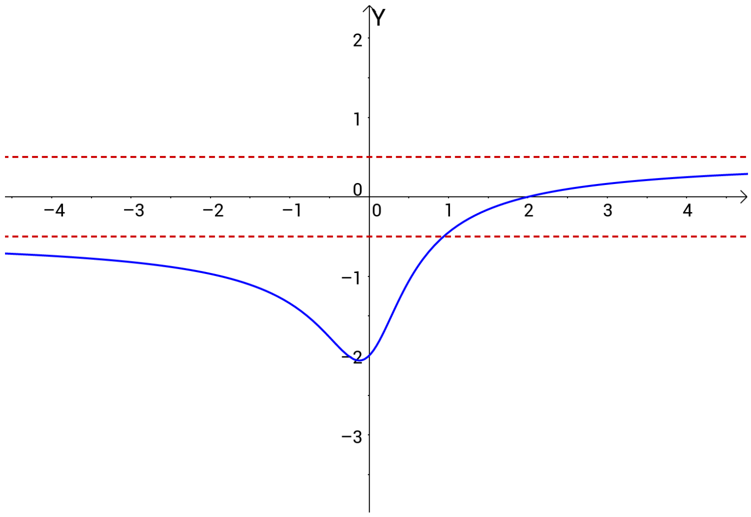
(xi) $f(x) = \dfrac{e^x}{x(x-1)}$:



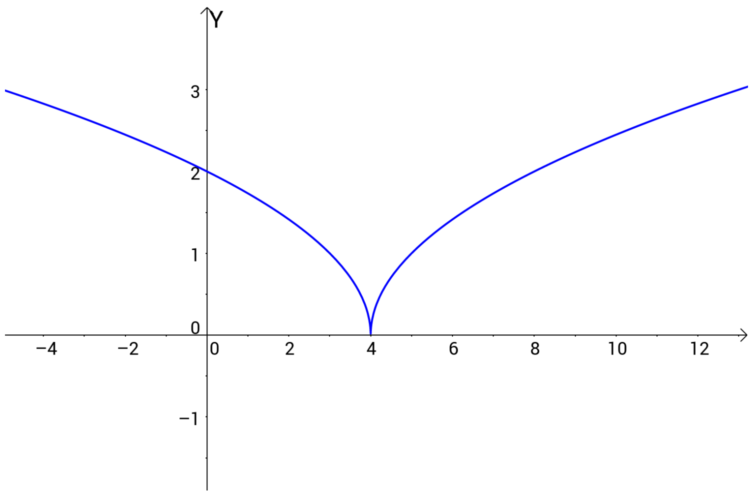
(xii) $f(x) = 2\sin x + \cos 2x$:



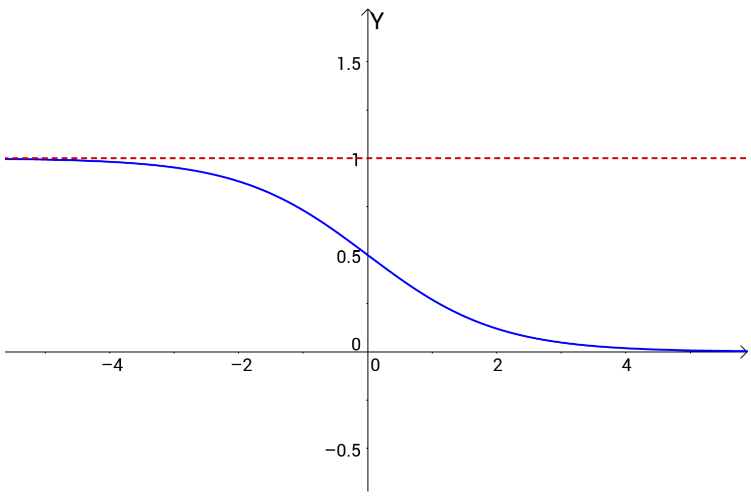
(xiii) $f(x) = \dfrac{x-2}{\sqrt{4x^2+1}}$:



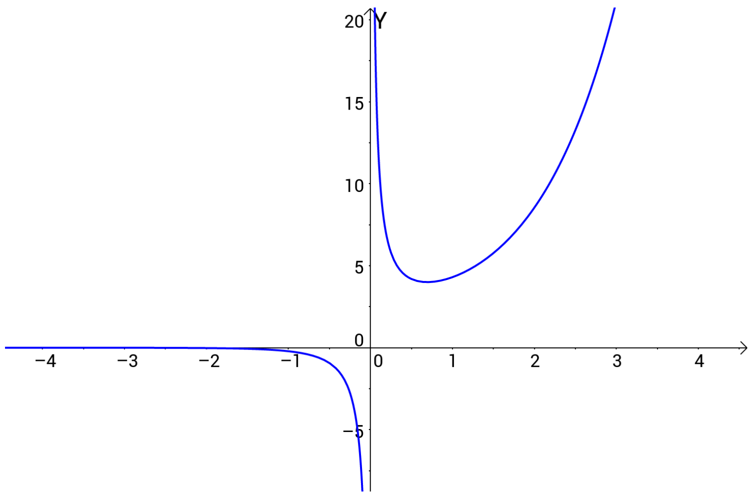
(xiv) $f(x) = \sqrt{|x-4|}$:



(xv) $f(x) = \dfrac{1}{1+e^x}$:



(xvi) $f(x) = \dfrac{e^{2x}}{e^x - 1}$:



(xvii) $f(x) = e^{-x}\sin x$: this function oscillates between $y = e^{-x}$ and $y = -e^{-x}$, crossing the X axis at $x = n\pi$, where $n \in \mathbb{Z}$.

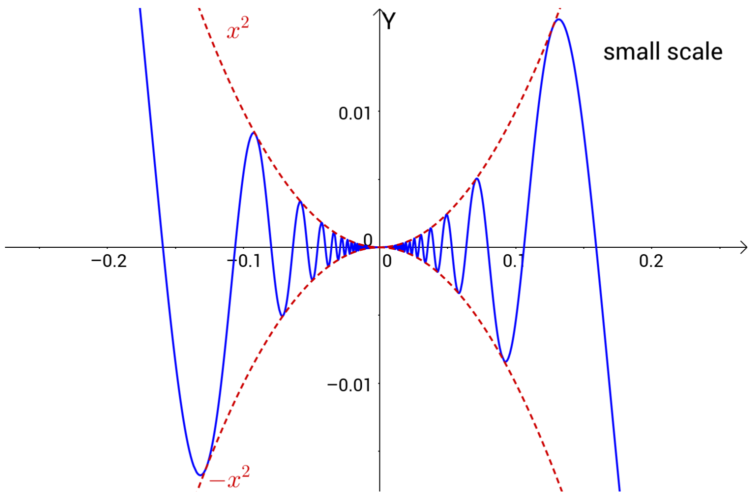
(xviii) $f(x) = x^2 \sin \dfrac{1}{x}$: this function has an oblique asymptote because

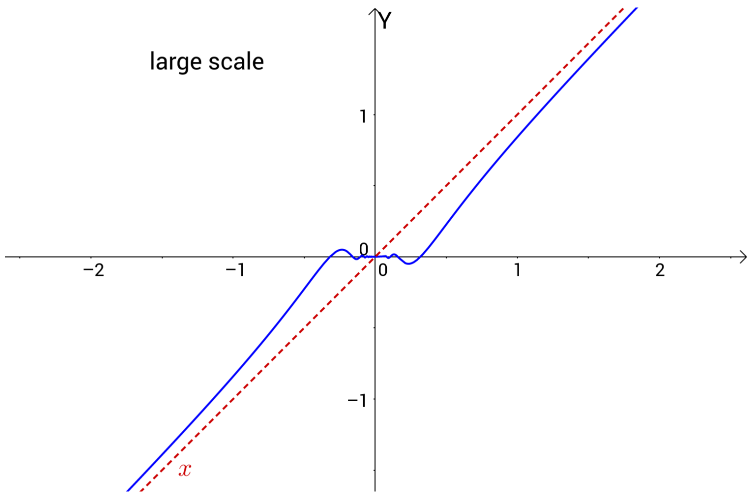
$$\sin\frac{1}{x} = \frac{1}{x} + o\left(\frac{1}{x^2}\right) \quad (x \longrightarrow \pm\infty)$$

(given that $\sin t = t + o(t^2)$ $(t \longrightarrow 0)$); hence

$$f(x) = x^2\left[\frac{1}{x} + o\left(\frac{1}{x^2}\right)\right] = x + o(1) \quad (x \longrightarrow \pm\infty).$$

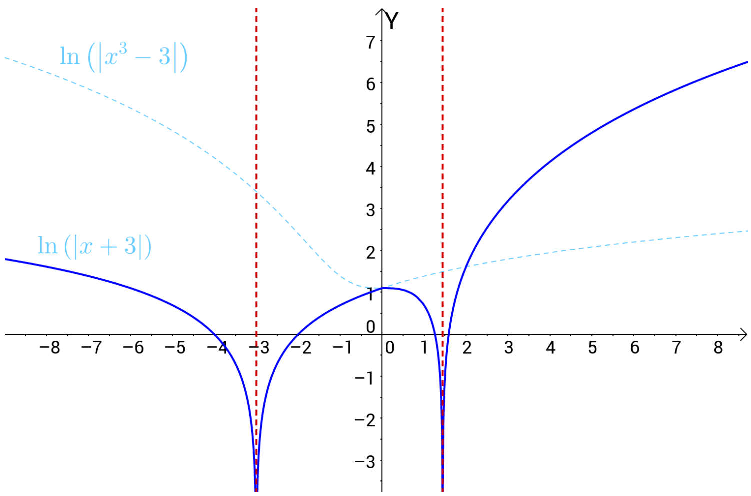
Therefore the function looks different on a small scale and on a large scale. On a small scale it is an oscillatory function between $-x^2$ and x^2 that crosses the X axis at $x = \pm\frac{1}{n\pi}$, for all $n \in \mathbb{Z} - \{0\}$; on a large scale it asymptotes to $y = x$:





Exercise 4.10

(i) $f(x) = \min\{\log|x^3 - 3|, \log|x + 3|\}$:



(ii) $f(x) = \dfrac{1}{|x| - 1} - \dfrac{1}{|x - 1|}$: this function has a different form for $x > 1$, for $0 < x < 1$ and for $x < 0$.
For $x > 1$

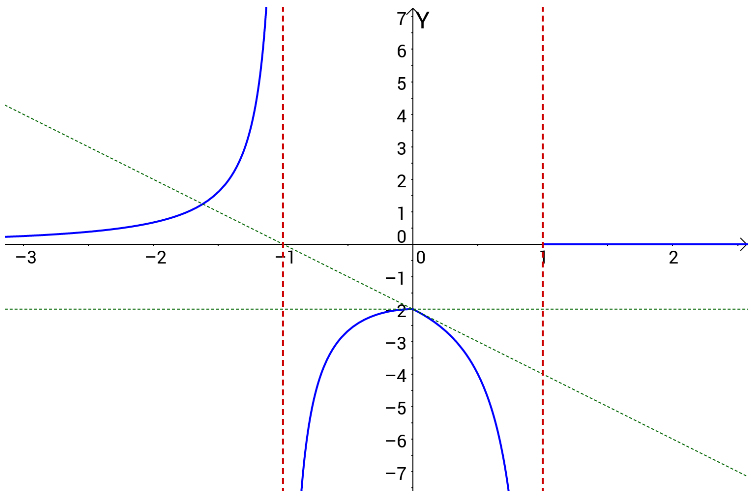
$$f(x) = \frac{1}{x-1} - \frac{1}{x-1} = 0.$$

For $0 < x < 1$ we have $|x - 1| = -(x - 1)$ so

$$f(x) = \frac{1}{x-1} + \frac{1}{x-1} = \frac{2}{x-1}.$$

For $x < 0$ we have $|x| - 1 = -(x + 1)$ and $|x - 1| = -(x - 1)$, so

$$f(x) = -\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}.$$



(iii) $f(x) = \dfrac{1}{1+|x|} - \dfrac{1}{1+|x-a|}$ $(a > 0)$: this function also has different definitions depending on whether $x > a$, $0 < x < a$, or $x < 0$.
For $x > a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+x-a} = \frac{-a}{(x+1)(x-a+1)},$$

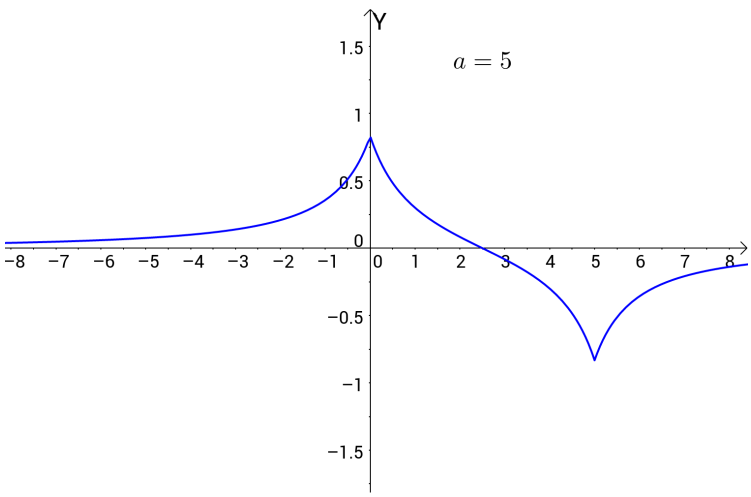
which has two vertical asymptotes, $x = -1$ and $x = a - 1$, both out of the region $x > a$.
For $0 < x < a$

$$f(x) = \frac{1}{1+x} - \frac{1}{1+a-x} = \frac{2x-a}{(x+1)(x-a-1)},$$

which again has two asymptotes, $x = -1$ and $x = a + 1$, both out of the region $0 < x < a$.
For $x < 0$

$$f(x) = \frac{1}{1-x} - \frac{1}{1+a-x} = \frac{a}{(x-1)(x-a-1)},$$

which also has two asymptotes, $x = 1$ and $x = a + 1$, both out of the region $x < 0$.
Here is a plot for $a = 5$ (which is generic):

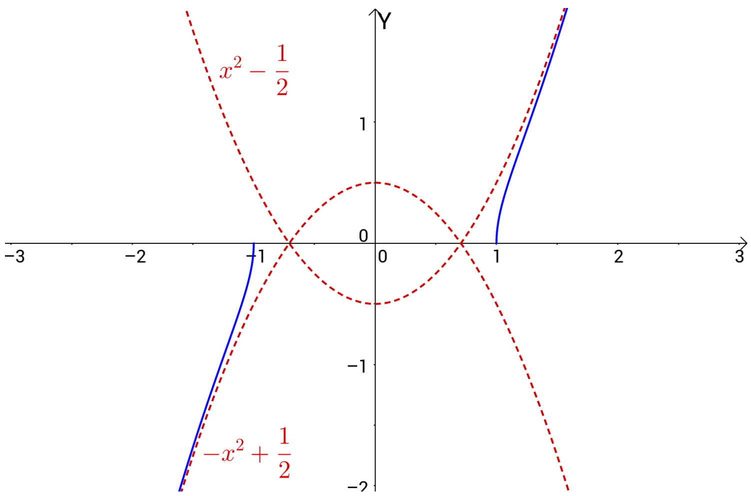


(iv) $f(x) = x\sqrt{x^2 - 1}$: notice that

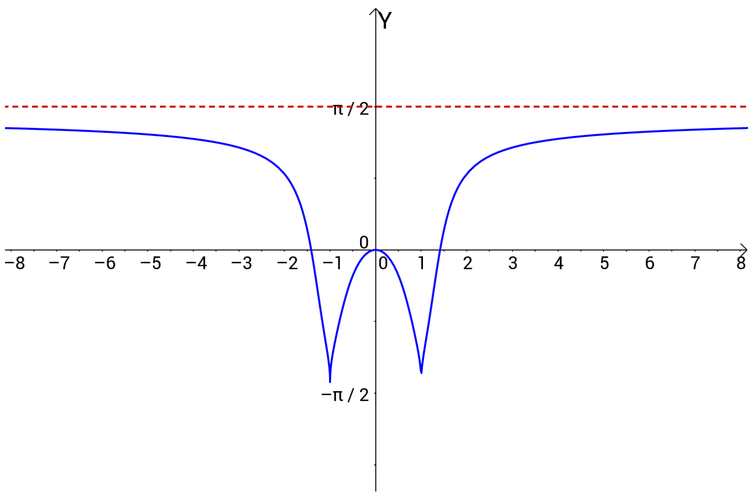
$$f(x) = x|x|\sqrt{1 - \frac{1}{x^2}},$$

and since $\sqrt{1-t} = 1 - t/2 + o(t)$ $(t \longrightarrow 0)$, when $x \longrightarrow \pm\infty$,

$$f(x) = x|x|\left[1 - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right] = x|x| - \frac{|x|}{2x} + o(1) = \begin{cases} x^2 - \frac{1}{2} + o(1) & (x \longrightarrow \infty), \\ -x^2 + \frac{1}{2} + o(1) & (x \longrightarrow -\infty). \end{cases}$$



(v) $f(x) = \arctan\log|x^2 - 1|$: when $x \longrightarrow \pm 1$ the logarithm diverges to $-\infty$, so $f(x) \longrightarrow -\pi/2$. In other words, even though the function is not well defined in $x = \pm 1$, at these two points it has an *avoidable* discontinuity which can be remedied by setting $f(\pm 1) = -\pi/2$. On the other hand, as $x \longrightarrow \pm\infty$ the logarithm diverges to ∞ and therefore $f(x) \longrightarrow \pi/2$.



(vi) $f(x) = 2\arctan x + \arcsin\left(\dfrac{2x}{1+x^2}\right)$: the domain of this function is $\mathbb{R}$ because so is the domain of $\arctan x$ and the argument of the arcsin is within $[-1, 1]$. To see this

$$\begin{aligned}
(x-1)^2 \ge 0 \quad &\Leftrightarrow \quad x^2 - 2x + 1 \ge 0 \quad \Leftrightarrow \quad x^2 + 1 \ge 2x \quad \Leftrightarrow \quad \frac{2x}{x^2+1} \le 1, \\
(x+1)^2 \ge 0 \quad &\Leftrightarrow \quad x^2 + 2x + 1 \ge 0 \quad \Leftrightarrow \quad x^2 + 1 \ge -2x \quad \Leftrightarrow \quad -\frac{2x}{x^2+1} \le 1 \\
&\Leftrightarrow \quad \frac{2x}{x^2+1} \ge -1.
\end{aligned}$$

if we calculate $f'(x)$, using the fact that

$$\left(\frac{2x}{1+x^2}\right)' = \frac{2(1+x^2) - (2x)^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2},$$

we obtain

$$f'(x) = \frac{2}{1+x^2} + \frac{1}{\sqrt{1 - \frac{4x^2}{(1+x^2)^2}}} \frac{2(1-x^2)}{(1+x^2)^2}.$$

But

$$1 - \frac{4x^2}{(1+x^2)^2} = \frac{1 + 2x^2 + x^4 - 4x^2}{(1+x^2)^2} = \frac{1 - 2x^2 + x^4}{(1+x^2)^2} = \frac{(1-x^2)^2}{(1+x^2)^2},$$

so

$$f'(x) = \frac{2}{1+x^2} + \frac{(1+x^2)}{|1-x^2|} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}\left[1 + \frac{1-x^2}{|1-x^2|}\right].$$

Now

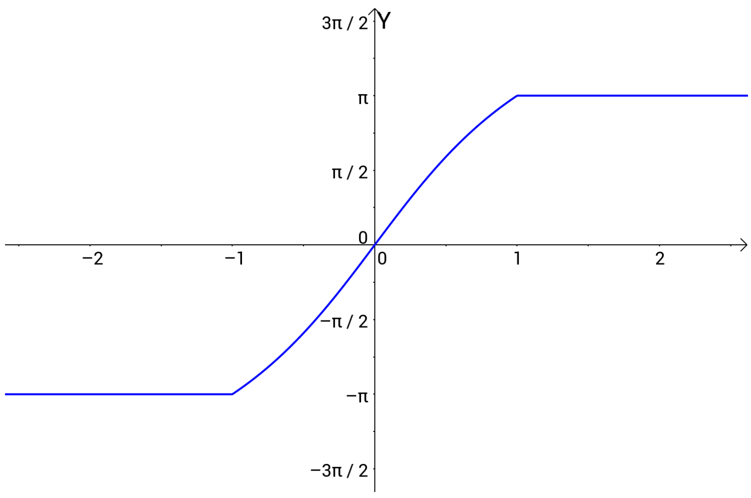
$$\frac{1-x^2}{|1-x^2|} = \begin{cases} 1, & |x| < 1, \\ -1, & |x| > 1, \end{cases}$$

therefore

$$f'(x) = \begin{cases} \dfrac{4}{1+x^2}, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Function $f(x)$ is thus constant if $|x| > 1$ and strictly increasing if $|x| < 1$. Besides, $f(x)$ is obviously continuous because so are all functions involved, so the constant values it takes for $x \ge 1$ and $x \le -1$ can be found as

$$f(\pm 1) = \pm 2 \arctan 1 + \pm \arcsin 1 = \pm\left(2 \cdot \frac{\pi}{4} + \frac{\pi}{2}\right) = \pm\pi.$$


Y
3π / 2
π
π / 2
0
−π / 2
−π
−3π / 2
−2
−1
0
1
2

A.5 Fundamental Theorem of Calculus

Exercise 5.1

(a) Changing $x = -t$,

$$I = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-t)\,dt = -\int_{-a}^{a} f(t)\,dt = -I \quad \Rightarrow \quad 2I = 0 \quad \Rightarrow \quad I = 0.$$

(b) Using the same change,

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} \underbrace{f(-t)}_{=f(t)}\,dt = 2\int_{0}^{a} f(x)\,dx.$$

(c) Changing $t = x - 8$,

$$\int_{6}^{10} \sin\left(\sin\left((x-8)^3\right)\right)dx = \int_{-2}^{2} \sin\left(\sin\left(t^3\right)\right)dt = 0$$

because the integrand is an odd function.

Exercise 5.2 We will approximate each integral using n equal subintervals and compare the results with the exact values.

(a) $\int_{-1}^{1}(1-x^2)\,dx$ with $n = 5$ subintervals.
First, divide the interval $[-1, 1]$ into $n = 5$ equal subintervals. The length of each subinterval is

$$w = \frac{1-(-1)}{5} = \frac{2}{5} = 0.4.$$

The approximation of the integral by rectangles is given by

$$T_5 = w\left(f(-1) + f(-0.6) + f(-0.2) + f(0.2) + f(0.6)\right) = 0.4{*}(0{+}0.64{+}0.84{+}0.84{+}0.64) = 1.28$$

The exact value of the integral is

$$\int_{-1}^{1}(1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3} \approx 1.333.$$

(b) $\int_{-1}^{2} e^{-x}\,dx$ with $n = 3$ subintervals.
The length of each subinterval is

$$h = \frac{2-(-1)}{3} = 1.$$

The approximation of the integral by rectangles is given by

$$T_3 = f(-1) + f(0) + f(1) = e^1 + 1 + e^{-1} = 4.086161269630487\ldots$$

The exact value of the integral is

$$\int_{-1}^{2} e^{-x}\,dx = [-e^{-x}]_{-1}^{2} = -(e^{-2} - e^1) = 2.5829465452224323\ldots$$

(c) $\int_{0}^{\pi} \sin x\,dx$ with $n = 4$ subintervals.

The length of each subinterval is

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}.$$

The approximation is

$$T_4 = \frac{\pi}{4}\left(f(0) + f(\pi/4) + f(\pi/2) + f(3\pi/4)\right) = \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{2}\right) = 1.8961188979370398\ldots$$

The exact value of the integral is

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2.$$

Exercise 5.3 We will calculate each integral by interpreting it as the signed area under the graph of the function. In each case, we recognize the geometric shape and apply the appropriate area formula.

(a) $\int_{-3}^{3} |x| \, dx$
The graph of $f(x) = |x|$ forms a "V" shape, symmetric about the y-axis. This is made of two triangles, each with base 3 and height 3. The area of one triangle is:

$$A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 3 \times 3 = 4.5.$$

Since the graph is symmetric, the total area is $2 \times 4.5 = 9$. Therefore, the value of the integral is:

$$\int_{-3}^{3} |x| \, dx = 9.$$

(b) $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$
The graph of $f(x) = \sqrt{9 - x^2}$ is a semicircle with radius 3 centered at the origin. The area of a full circle is $A = \pi r^2 = \pi(3^2) = 9\pi$, and the area of the upper half of the circle (which is the region under the graph) is:

$$A = \frac{1}{2} \times 9\pi = \frac{9\pi}{2}.$$

Therefore, the value of the integral is:

$$\int_{-3}^{3} \sqrt{9 - x^2} \, dx = \frac{9\pi}{2}.$$

(c) $\int_{2}^{5} \left(\frac{x}{2} - 4\right) \, dx$
The graph of $f(x) = \frac{x}{2} - 4$ is a straight line, and we are integrating from $x = 2$ to $x = 5$. This region forms a trapezoid. The height of the trapezoid is the difference between the x-coordinates: $h = 5 - 2 = 3$. The function values at the endpoints are:

$$f(2) = \frac{2}{2} - 4 = -3, \quad f(5) = \frac{5}{2} - 4 = -\frac{3}{2}.$$

The area of the trapezoid is:

$$A = \frac{1}{2} \times (b_1 + b_2) \times h = \frac{1}{2} \times (-3 + -\frac{3}{2}) \times 3 = \frac{1}{2} \times -\frac{9}{2} \times 3 = -\frac{27}{4}.$$

Therefore, the value of the integral is:

$$\int_2^5 \left(\frac{x}{2} - 4\right) dx = -\frac{27}{4}.$$

(d) $\int_{-1}^{2} (2 - |x|)\, dx$
The graph of $f(x) = 2 - |x|$ forms a trapezoid from $x = -1$ to $x = 0$, with area $A_1 = 1/2$ and a triangle form $x = 0$ to $x = 2$, with area $A_2 = 2$. The value of the integral is then $A = 5/2$.

Exercise 5.4 (i) For $x < 1/2$,

$$F(x) = \int_{-1}^{x} \left(\frac{1}{2} - t\right) dt = \frac{2 + x - x^2}{2} = \frac{(2 - x)(1 + x)}{2}$$

For $x \geq 1/2$,

$$F(x) = \int_{-1}^{1/2} \left(\frac{1}{2} - t\right) dt + \int_{1/2}^{x} \left(t - \frac{1}{2}\right) dt = \frac{9}{4} + \frac{(x - 2)(1 + x)}{2}.$$

(ii) For $x < 0$,

$$F(x) = \int_{-1}^{x} (-1)\, dt = -1 - x.$$

For $x \geq 0$,

$$F(x) = \int_{-1}^{0} (-1)\, dt + \int_{0}^{x} dt = -1 + x.$$

Thus, $F(x) = |x| - 1$.

(iii) For $x < 0$,

$$F(x) = \int_{-1}^{x} t^2\, dt = \frac{x^3 + 1}{3}.$$

For $x \geq 0$,

$$F(x) = \int_{-1}^{0} t^2\, dt + \int_{0}^{x} (t^2 - 1) dt = \int_{-1}^{x} t^2\, dt - \int_{0}^{x} dt = \frac{x^3 + 1}{3} - x = \frac{x^3 - 3x + 1}{3}.$$

(iv) For $x \leq 0$,

$$F(x) = \int_{-1}^{x} dt = x + 1.$$

For $x > 0$,

$$F(x) = \int_{-1}^{0} dt + \int_{0}^{x} (t + 1) dt = \int_{-1}^{x} dt + \int_{0}^{x} t\, dt = \frac{x^2}{2} + x + 1.$$

(v) For $x \leq -1/2$,

$$F(x) = \int_{-1}^{x} (1 + t)\, dt = \frac{(1 + x)^2}{2}.$$

For $-1/2 < x < 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1 + t)\, dt + \frac{1}{2} \int_{-1/2}^{x} dt = \frac{1}{8} + \frac{2x + 1}{4} = \frac{4x + 3}{8}.$$

For $x \geq 1/2$,

$$F(x) = \int_{-1}^{-1/2} (1+t)\,dt + \frac{1}{2}\int_{-1/2}^{1/2} dt + \int_{1/2}^{x} (1-t)\,dt = \frac{3}{4} - \frac{(1-x)^2}{2}.$$

Exercise 5.5

(i) $F'(x) = \dfrac{3e^{x^3} - 2e^{x^2}}{x}$.

(ii) $F'(x) = \dfrac{6x^2}{1+\sin^2(x^3)}$.

(iii) $F'(x) = 2x\displaystyle\int_0^x f(t)\,dt + x^2 f(x)$.

Exercise 5.6 $f'(x) = e^{-(x-1)^2} - e^{-2(x-1)}$, so $f'(x) = 0$ when $(x-1)^2 = 2(x-1)$, i.e., when $x = 1$ or $x = 3$. Between those two values $(x-1)^2 < 2(x-1)$, and for $x > 3$ the opposite holds. Therefore $f'(x) > 0$ for $1 < x < 3$ and $f'(x) < 0$ for $x > 3$. Thus there is a local maximum at $x = 3$ —which is the absolute maximum. To obtain the absolute minimum we need to obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\int_0^{x-1} e^{-t^2}\,dt - \int_0^{x-1} e^{-2t}\,dt\right) = \frac{\sqrt{\pi}}{2} - \lim_{x\to\infty}\frac{1}{2}\left(1 - e^{-2(x-1)}\right) = \frac{\sqrt{\pi}-1}{2} > 0.$$

Since $f(1) = 0$, the absolute minimum is reached at $x = 1$.

Exercise 5.7 Function $f(x) = \displaystyle\int_0^x e^{t^2}\,dt - 1$ is an increasing function because $f'(x) = e^{x^2} > 0$. Further $f(0) = -1$. On the other hand, $e^{t^2} > 1$ for all $t > 0$, so

$$f(1) = \int_0^1 e^{t^2}\,dt - 1 > \int_0^1 dt - 1 = 0.$$

Therefore $f(x) = 0$ has a unique solution in $(0, 1)$.

Exercise 5.8 $F(x)$ is a continuous function (is the difference of two integrals) in $[0, 1]$. On the other hand,

$$F(0) = 2\cancel{\int_0^0 f(t)\,dt} - \int_0^1 f(t)\,dt = -\int_0^1 f(t)\,dt < 0$$

(it is negative because $f(x) > 0$ in $[0, 1]$, therefore the integral is positive), and

$$F(1) = 2\int_0^1 f(t)\,dt - \cancel{\int_1^1 f(t)\,dt} = 2\int_0^1 f(t)\,dt > 0$$

(it is positive for the same reason). Since $F(x)$ has opposite signs at the extremes of the interval it must be zero somewhere in between. Thus, the equation $F(x) = 0$ has at least one solution. To see that there are no more solutions we differentiate

$$F'(x) = 2f(x) - f(x)(-1) = 3f(x) > 0.$$

Therefore $F(x)$ increases monotonically in $[0, 1]$, hence can be zero only once in the interval.

Exercise 5.9

$$f'(x) = \frac{1}{a^2+x^2} - \frac{1}{x^2}\frac{1}{a^2+1/x^2} = \frac{1}{a^2+x^2} - \frac{1}{a^2x^2+1},$$

so in order to have $f'(x) = 0$ for any x we need $a = \pm 1$.

Exercise 5.10

(a) $dt = 2\sin\theta\cos\theta\, d\theta = \sin 2\theta\, d\theta$, therefore

$$I = \int_0^1 \arcsin\sqrt{t}\, dt = \int_0^{\pi/2} \arcsin(\sin\theta)\,\sin 2\theta\, d\theta = \int_0^{\pi/2} \theta\sin 2\theta\, d\theta.$$

We can now integrate by parts, where $u = \theta$ and $v' = \sin 2\theta$, and then

$$I = -\frac{\theta}{2}\cos 2\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\, d\theta = \frac{\pi}{4} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} = \frac{\pi}{4} + 0.$$

Thus

$$\int_0^1 \arcsin\sqrt{t}\, dt = \frac{\pi}{4}.$$

(b) Differentiating,

$$f'(x) = 2\sin x\cos x\arcsin(\sin x) - 2\cos x\sin x\arccos(\cos x) = x\sin 2x - x\sin 2x = 0.$$

Therefore $f(x)$ is constant.

(c) We can calculate c by substituting any value of x, for instance $x = \pi/2$. Then

$$c = f(\pi/2) = \int_0^1 \arcsin\sqrt{t}\, dt + \int_0^0 \arccos\sqrt{t}\, dt = \int_0^1 \arcsin\sqrt{t}\, dt.$$

But this is precisely the integral we have obtained in (a), so $c = \pi/4$.

Exercise 5.11

(a) With this change of variables the limits remain the same, so

$$I = \int_0^{\pi} x f(\sin x)\, dx = \int_0^{\pi} (\pi - y) f(\sin(\pi - y))\, dy.$$

But since $\sin(\pi - y) = \sin y$, we have

$$I = \int_0^{\pi} (\pi - y) f(\sin y)\, dy = \pi\int_0^{\pi} f(\sin y)\, dy - I.$$

Thus

$$I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$$

(b) Since

$$\frac{\sin x}{1+\cos^2 x} = \frac{\sin x}{2-\sin^2 x} = f(\sin x),$$

we are in the situation described in the previous item. Hence

$$
\begin{aligned}
I &= \int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_0^\pi \frac{(\cos x)'}{1+\cos^2 x}\,dx \\
&= -\frac{\pi}{2}\arctan(\cos x)\Big|_0^\pi = -\frac{\pi}{2}(-2\arctan 1) = \frac{\pi^2}{4}.
\end{aligned}
$$

Exercise 5.12 (a) To find $N(t)$, we need to integrate the differential equation:

$$\frac{dN}{dt} = e^{-t}.$$

Integrating both sides with respect to t, and knowing that $N(0) = 100$, we have:

$$N(t) - N(0) = \int_0^t e^{-t}\,dt = 1 - e^{-t} \implies = N(t) = 101 - e^{-t}.$$

(b) To compute the cumulative change in population size between $t = 0$ and $t = 5$, we need to evaluate the change in $N(t)$ over this interval:

$$\Delta N = N(5) - N(0).$$

We already know $N(0) = 100$. Now, calculate $N(5)$:

$$N(5) = -e^{-5} + 101.$$

Using $e^{-5} \approx 0.0067$, we have

$$N(5) \approx -0.0067 + 101 = 100.9933.$$

Therefore, the cumulative change in population size is:

$$\Delta N = 100.9933 - 100 = 0.9933.$$

Hence, the cumulative change in population size between $t = 0$ and $t = 5$ is approximately 0.9933.

Exercise 5.13 We are given the velocity of a particle moving along the x-axis as

$$v(t) = -(t-2)^2 + 1$$

for $0 \le t \le 5$. We also know that the particle starts at the origin at time $t = 0$. We will analyze the motion of the particle step by step.

(a) The particle moves to the right when its velocity $v(t)$ is positive, and it moves to the left when $v(t)$ is negative. To find when this happens, we first examine the graph and behavior of $v(t)$.
The velocity function is:

$$v(t) = -(t-2)^2 + 1.$$

This is a downward-opening parabola with its vertex at $t = 2$ and maximum value $v(2) = 1$. The roots of $v(t) = 0$ occur when:

$$-(t-2)^2 + 1 = 0 \quad \Rightarrow \quad (t-2)^2 = 1 \quad \Rightarrow \quad t - 2 = \pm 1,$$

which gives:

$$t = 1 \quad \text{and} \quad t = 3.$$

Therefore, $v(t) = 0$ at $t = 1$ and $t = 3$.
Now, examine the sign of $v(t)$:
- For $0 \le t < 1$, $v(t) > 0$, so the particle moves to the right.
- For $1 < t < 3$, $v(t) < 0$, so the particle moves to the left.
- For $3 < t \le 5$, $v(t) > 0$, so the particle moves to the right again.

Thus, the particle moves to the right on the intervals $0 \le t < 1$ and $3 < t \le 5$, and it moves to the left on the interval $1 < t < 3$.

(b) The position $s(t)$ of the particle is the integral of its velocity $v(t)$ with respect to time. Since the particle is at the origin at $t = 0$, we have the initial condition $s(0) = 0$. Thus, we find $s(t)$ by integrating $v(t)$:

$$s(t) = s(0) + \int_0^t v(\tau)\, d\tau = \int_0^t \left(-(\tau-2)^2 + 1\right)\, d\tau.$$

Let's compute this integral:

$$s(t) = \int_0^t \left(-(\tau-2)^2 + 1\right)\, d\tau = \int_0^t -(\tau^2 - 4\tau + 4) + 1\, d\tau.$$

Simplifying:

$$s(t) = \int_0^t (-\tau^2 + 4\tau - 3)\, d\tau = \left[-\frac{\tau^3}{3} + 2\tau^2 - 3\tau\right]_0^t.$$

Evaluating the definite integral:

$$s(t) = -\frac{t^3}{3} + 2t^2 - 3t.$$

The location $s(t)$ represents the net area under the velocity curve $v(t)$ from $t = 0$ to t. Positive areas (where $v(t) > 0$) correspond to the particle moving to the right, while negative areas (where $v(t) < 0$) correspond to the particle moving to the left. The function $s(t)$ gives the cumulative displacement of the particle along the x-axis.

Exercise 5.14 We are given the average daily temperature function:

$$T(t) = 57.5 - 22.5\cos(2\pi t),$$

where t represents the fraction of the year that has elapsed since January 1. We will solve the parts of the problem step by step.

(a) To find the average temperature over the year, we need to compute the average value of the function $T(t)$ over the interval $0 \le t \le 1$. The average value of a function over an interval $[a, b]$ is given by:

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(t)\, dt.$$

Here, $a = 0$, $b = 1$, and $f(t) = T(t)$. Thus, the average temperature is:

$$\text{Average temperature} = \int_0^1 T(t)\, dt = \int_0^1 \left(57.5 - 22.5\cos(2\pi t)\right)\, dt.$$

We can split the integral:

$$\int_0^1 (57.5 - 22.5\cos(2\pi t))\ dt = 57.5\int_0^1 1\,dt - 22.5\int_0^1 \cos(2\pi t)\,dt.$$

The first integral is straightforward:

$$\int_0^1 1\,dt = 1.$$

For the second integral,

$$\int_0^1 \cos(2\pi t)\,dt = \frac{1}{2\pi}\sin(2\pi t)\Big|_0^1 = 0.$$

Thus, the average temperature is:

$$\text{Average temperature} = 57.5 \times 1 - 22.5 \times 0 = 57.5.$$

(b) We can observe that the function $T(t) = 57.5 - 22.5\cos(2\pi t)$ consists of a constant term 57.5 and an oscillating term $-22.5\cos(2\pi t)$. Since $\cos(2\pi t)$ oscillates symmetrically about zero over the interval $[0, 1]$, its average value is zero. Therefore, the average value of $T(t)$ is simply the constant 57.5, which is the baseline temperature. This reasoning allows us to determine the average temperature without performing any integration.

(c) Summer corresponds to the interval $0.47 \le t \le 0.73$. To find the average temperature during the summer, we use the formula for the average value of the function over this interval:

$$\text{Average summer temperature} = \frac{1}{0.73 - 0.47}\int_{0.47}^{0.73} T(t)\,dt.$$

First, simplify the factor:

$$\frac{1}{0.73 - 0.47} = \frac{1}{0.26}.$$

Now, compute the integral:

$$\int_{0.47}^{0.73} T(t)\,dt = \int_{0.47}^{0.73} (57.5 - 22.5\cos(2\pi t))\ dt.$$

Again, we split the integral:

$$\int_{0.47}^{0.73} (57.5 - 22.5\cos(2\pi t))\ dt = 57.5\int_{0.47}^{0.73} 1\,dt - 22.5\int_{0.47}^{0.73} \cos(2\pi t)\,dt.$$

The first integral is straightforward:

$$\int_{0.47}^{0.73} 1\,dt = 0.26.$$

For the second integral, we need to compute:

$$\int_{0.47}^{0.73} \cos(2\pi t)\,dt.$$

The antiderivative of $\cos(2\pi t)$ is $\frac{1}{2\pi}\sin(2\pi t)$. Evaluating this from $t = 0.47$ to $t = 0.73$:

$$\frac{1}{2\pi}\left(\sin(2\pi \times 0.73) - \sin(2\pi \times 0.47)\right).$$

Using a calculator, we find:

$$\sin(2\pi \times 0.73) \approx -0.5878, \quad \sin(2\pi \times 0.47) \approx 0.5878.$$

Therefore:

$$\int_{0.47}^{0.73} \cos(2\pi t)\, dt = \frac{1}{2\pi}(-0.5878 - 0.5878) = \frac{-1.1756}{2\pi}.$$

Now, putting everything together, we find:

$$\text{Average summer temperature} = \frac{1}{0.26}\left(57.5 \times 0.26 - 22.5 \times \frac{-1.1756}{2\pi}\right).$$

Using $\pi \approx 3.1416$, this simplifies to:

$$\text{Average summer temperature} \approx 57.5 - 4.21 = 53.29°\text{F}.$$

A.6 Differential Equations

Exercise 6.1 (a) $\dot{x} = 4x^2 - 16$

- Rewrite the equation: $\dot{x} = 4(x^2 - 4) = 4(x-2)(x+2)$.
- Fixed points: $x = 2$ and $x = -2$.
- Stability: $x = -2$ is stable, and $x = 2$ is unstable.
- Graph of $x(t)$: If $x_0 < 2$ then the trajectories approach 2, and if $x_0 > 2$ they diverge to ∞.

(b) $\dot{x} = 1 - x^{14}$

- Fixed points: $x = \pm 1$.
- Stability: $x = -1$ is unstable, $x = 1$ is stable.
- Graph of $x(t)$: If $x_0 > -1$ then the trajectories approach 1, and if $x_0 < -1$ they diverge to ∞.

(c) $\dot{x} = x - x^3$

- Rewrite the equation: $\dot{x} = x(1 - x^2) = x(1-x)(1+x)$.
- Fixed points: $x = -1$, $x = 0$, and $x = 1$.
- Stability: $x = -1$ and $x = 1$ are stable, while $x = 0$ is unstable.
- Graph of $x(t)$: If $x_0 < 0$ the system approaches -1, whereas if $x_0 > 0$ the system approaches 1.

(d) $\dot{x} = e^{-t} \sin x$

- Fixed points: $x = n\pi$ for integers n.
- Stability: Here $\dot{x} = f(x,t)$ and the stability comes from $\partial f / \partial x$ the partial derivative of f with respect to x considering t a constant. Since $e^{-t} > 0$ for all t, the stability of the system is the same as that of $\dot{x} = \sin x$.
- Graph of $x(t)$: Because e^{-t} decays so quickly, even though the only fixed points are $x^* = k\pi$ for $k \in \mathbb{Z}$, the trajectories remain very close to the initial condition for long periods of time (see Figure A.1).

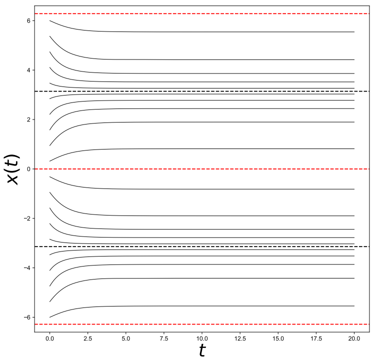


Figure A.1: Trajectories for the differential equation $\dot{x} = e^{-t} \sin x$.

(e) $\dot{x} = 1 + \cos x$

- Fixed points: $x = \pi + 2n\pi$ for integers n.
- Stability: The fixed points are all "mixed": the derivative is positive to both sides, which means that initial conditions starting below the fixed points tend towards it, while those starting above it are driven away from it.
- Graph of $x(t)$: The derivative is always non-negative, which means that $x(t)$ is always growing. Initial conditions between $\pi + 2n\pi$ and $\pi + 2(n+1)\pi$ will grow towards $x^* = \pi + 2(n+1)\pi$.

(f) $\dot{x} = 1 - e^{\cos x}$

- Fixed points: $x = \frac{\pi}{2} + n\pi$ for integers n.
- Stability: If n is odd in the set of fixed points above, the point is stable. Otherwise, it is unstable.
- Graph of $x(t)$: for n odd, all initial conditions between $\frac{\pi}{2} + (n-1)\pi$ and $\frac{\pi}{2} + (n+1)\pi$ tend toward $\frac{\pi}{2} + n\pi$.

(g) $\dot{x} = e^x - \cos x$

- Fixed points: Fixed points occur where $e^x = \cos x$. The exact values of x cannot be found explicitly, but graphically we see there are infinitely many points between $-\infty$ and 0, as e^x is between 0 and 1 in this interval, and $\cos x$ crosses the x axis infinitely many times.

- Stability: Stability can be assessed qualitatively by observing the slopes of e^x and $\cos x$.
- Graph of $x(t)$: similar to the previous case.

Exercise 6.2 (a) Here we show how to obtain the given formula (you don't have to do this! but in case you're interested in how to do it). Dividing both sides by m, we get:

$$\frac{dv}{dt} = g - \frac{k}{m}v^2.$$

This is a separable differential equation, so we can write:

$$\frac{dv/dt}{g - \frac{k}{m}v^2} = 1.$$

We can decompose the fraction in the left-hand side into partial fractions as follows:

$$\begin{aligned}\frac{1}{g - \frac{k}{m}v^2} &= \frac{1}{2\sqrt{g}}\left[\frac{1}{\sqrt{\frac{k}{m}}v + \sqrt{g}} - \frac{1}{\sqrt{\frac{k}{m}}v - \sqrt{g}}\right] \\ &= \frac{1}{2}\sqrt{\frac{mk}{g}}\left[\frac{1}{\sqrt{\frac{mg}{k}} + v} + \frac{1}{\sqrt{\frac{mg}{k}} - v}\right].\end{aligned}$$

Writing $c = \sqrt{\frac{mg}{k}}$, we can express the left-hand side as:

$$\int \frac{dv/dt}{g - \frac{k}{m}v^2} = \frac{c}{2g}\log\left(\frac{c+v}{c-v}\right).$$

The right-hand side is just t, so we have:

$$\frac{c}{2g}\log\left(\frac{c+v}{c-v}\right) = t + C.$$

Using the initial condition $v(0) = 0$, we can solve for C:

$$\frac{c}{2g}\log\left(\frac{c}{c}\right) = 0 \quad \Rightarrow \quad C = 0.$$

Thus, the solution becomes:

$$\frac{c}{2g}\log\left(\frac{c+v}{c-v}\right) = t.$$

Multiplying both sides by $2g/c$ and exponentiating, we get

$$\frac{c+v}{c-v} = e^{2gt/c} \implies c + v = e^{2gt/c}c - e^{2gt/c}v$$

Solving for $v(t)$, we obtain:

$$v(t) = c\frac{1-e^{-2gt/c}}{1+e^{-2gt/c}} = \sqrt{\frac{mg}{k}}\frac{1-e^{-2\sqrt{gk/mt}}}{1+e^{-2\sqrt{gk/mt}}}.$$

To find the terminal velocity, we take the limit as $t \longrightarrow \infty$ of $v(t)$. Since $\exp(-2\sqrt{\frac{gk}{m}}\,t) \longrightarrow 0$ as $t \longrightarrow \infty$, we have:

$$v_\infty = \lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}}.$$

Thus, the terminal velocity is $v_\infty = \sqrt{\frac{mg}{k}}$.

(b) As we can see in Figure A.2, $\dot{v} = 0$ only when $v^* = \sqrt{\frac{mg}{k}}$, the only fixed point of the dynamics. If $v > v^*$ then $\dot{v} < 0$, and if $v < v^*$, then $\dot{v} > 0$, so this makes v^* a stable fixed point.

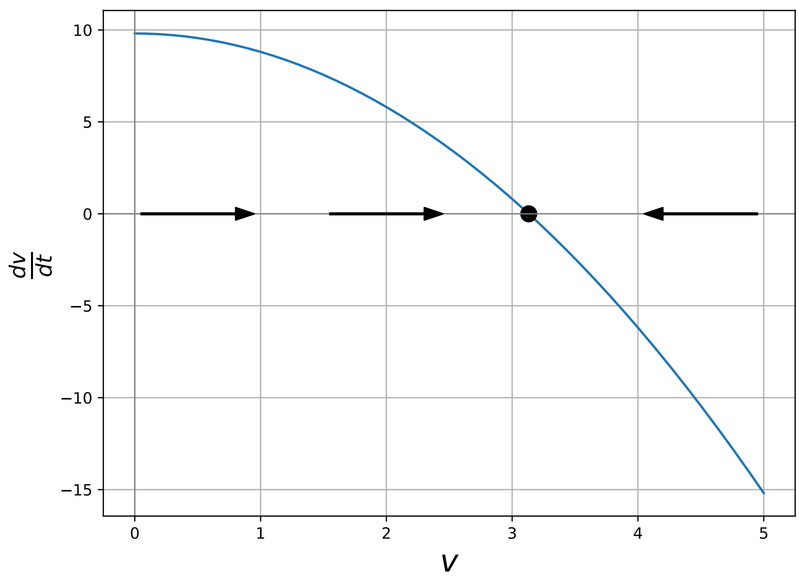


Figure A.2: Phase protrait of the differential equation $\dot{v} = g - k/mv^2$, showing the stable fixed point $v = \sqrt{\frac{mg}{k}}$.

Exercise 6.3 (a) The fixed points satisfy $x(k_1a - k_{-1}x) = 0$ and therefore we have $x_1^* = 0$ and $x_2^* = ak - 1/k_{-1}$. It is not hard to see that x_1^* is unstable and x_2^* is stable (this system is mathematically identical to the logistic equation).

(b) The trajectories can be seen in Figure A.3. They are very similar to those in the logistic equation (Figure 6.4).

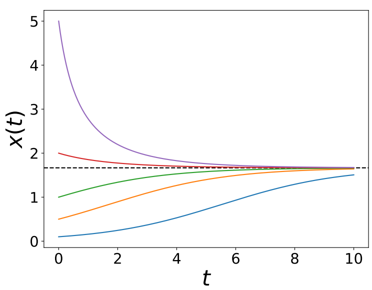


Figure A.3: Trajectories for the differential equation $\dot{x} = k_1ax - k_{-1}x^2$.

Exercise 6.4 (a) a is the growth rate of the cancer, and b is the inverse of the carrying capacity.

(b) There are two fixed points: $N_1^* = 0$ and $N_2^* = 1/b$. The trajectories are very similar to those shown in Figure A.3.

(c) If $f(N) = -aN\log(bN)$ then $f'(N) = -a(1+\log(bN))$. The derivative does not exist when $N =$, but it's clear that $\lim_{N\to 0} f'(N) > 0$ and so $N = 0$ is an unstable fixed point. However, $f'(1/b) = -a < 0$ and so $N = 1/b$ is a stable fixed point.

Exercise 6.5 (a) The equation is a parabola with a maximum at an intermediate N provided that $a, b, r > 0$. If $r > ab^2$ the intercept is positive, which means growth is always positive. If $r < ab^2$ then the intercept is negative, which means that for some N growth will be negative.

(b) There are three fixed points: $N_1^* = 0$, $N_2^* = b - \sqrt{r/a}$ and $N_3^* = b + \sqrt{r/a}$. The derivative of $f(N) = N(r - a(N-b)^2)$ is

$$f'(N) = r - ab^2 + 4abN - 3aN^2.,$$

We have

$$\begin{aligned} f'(N_1^*) &= r - ab^2, \\ f'(N_2^*) &= -2\sqrt{r}\left(\sqrt{r} - b\sqrt{a}\right) = -2\sqrt{r}\frac{r - ab^2}{\sqrt{r} + b\sqrt{a}}, \\ f'(N_3^*) &= -2\sqrt{r}\left(\sqrt{r} + 2b\sqrt{a}\right) < 0. \end{aligned}$$

Note that $f'(N_3^*)$ is always negative (and hence the point is always stable), but the behavior of the other two points changes whether $r > ab^2$ or $r < ab^2$. In the first case, the growth at $N = 0$ is positive (check $f(N)$) and we have that N_1^* is unstable and N_2^* is stable, but note that $N_2^* < 0$ in this case and therefore it will never be reached if $N(0) > 0$, which is the realistic scenario. This is called "weak" Allee effect. In this case, the trajectories are very similar to those in the logistic equation, only that the growth when N is small is slower.

However, if $r < ab^2$, then the growth when $N = 0$ is negative. N_1^* becomes stable and N_2^* (which is now positive) turns unstable. In this case, the trajectories are very different depending on the initial conditions: if $N(0) < N_2^*$, then the population becomes extinct, whereas if $N(0) > N_2^*$, the population reaches the carrying capacity. This is called a "strong" Allee effect.

The two cases are shown in Figure A.4.

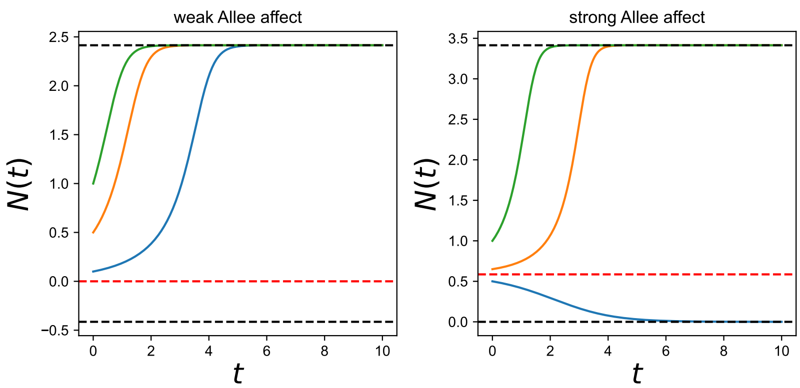


Figure A.4: Trajectories for the differential equation $\dot{N} = N(r - a(N-b)^2)$, with $r = 1, a = 0.5, b = 1$ (left, weak Allee effect) and $r = 1, a = 0.5, b = 2$ (right, strong Allee effect).

Exercise 6.6 (a) Two fixed points, $x = 0$ and $x = 1$. The derivatibe of $f(x)$ is $f'(x) = 1 - 2x$ which is positive for $x = 0$ and negative for $x = 1$.

(b) Two fixed points, $x = 0$ and $x = 1/2$. The derivatibe of $f(x)$ is $f'(x) = -1 + 4x$ which is negative for $x = 0$ and positive for $x = 1/2$.

(c) Infinite fixed points at $x = k\pi$, with $k \in \mathbb{Z}$. However, $f'(x) = 1 + \tan^2 x > 0$ and so every point is unstable.

(d) Three fixed points, at $x = 0$ and $x = \pm\sqrt{6}$. The derivative of $f(x)$ is $f'(x) = 12x - 4x^3$ which is 0 at $x = 0$, positive at $-\sqrt{6}$ (unstable fixed point) and negative at $x = \sqrt{6}$ (stable fixed point). Graphical analysis shows that $x = 0$ is a mixed fixed point: trajectories starting between $-\sqrt{6}$ and 0 are drawn towards 0, whereas trajectories starting at $x > 0$ are drawn towards $\sqrt{6}$.

(e) One fixed point at $x = 0$. The derivative of $f(x)$ is $f'(x) = e^{-x}$, which is positive at $x = 0$.

(f) One fixed point at $x = 1$, the derivative of $f(x)$ is $f'(x) = 1/x$, which is positive at $x = 1$.

(g) If $a > 0$, there are three fixed points: $x = 0$, $x = \pm\sqrt{a}$. The derivative of $f(x)$ is $f'(x) = a - 3x^2$, which is positive at $x = 0$ and negative at both $x = \pm\sqrt{a}$ (stable fixed points). If $a = 0$ there is only one fixed point at $x = 0$. The derivative is zero at that point, but a graphical analysis shows that it is a stable point. Finally, if $a < 0$ there is only one fixed point at $x = 0$ but now the derivative $f'(x) = a - 3x^2$ is negative at $x = 0$, and so the point is stable.

A.7 Linear Functions of Several Variables

Exercise 7.1 Since $\mathbf{x} = (x_1, x_2) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, we have (because T is linear) that

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{pmatrix}2\\5\end{pmatrix} + x_2\begin{pmatrix}-1\\6\end{pmatrix} = \begin{pmatrix}2x_1 - x_2\\5x_1 + 6x_2\end{pmatrix}$$

In particular,

$$T(5,3) = \begin{pmatrix}7\\43\end{pmatrix}$$

Exercise 7.2 We will write the computations on the augmented matrix.

1. $$\begin{pmatrix}2 & 1 & 6\\1 & -4 & -4\end{pmatrix} \sim \begin{pmatrix}1 & -4 & -4\\2 & 1 & 6\end{pmatrix} \sim \begin{pmatrix}1 & -4 & -4\\0 & 9 & 14\end{pmatrix} \sim \begin{pmatrix}1 & -4 & -4\\0 & 1 & \frac{14}{9}\end{pmatrix} \sim \begin{pmatrix}1 & 0 & \frac{20}{9}\\0 & 1 & \frac{14}{9}\end{pmatrix}$$
   Thus, $x = 20/9$ and $y = 14/9$.
2. $$\begin{pmatrix}5 & 2 & 8\\-1 & 3 & 9\end{pmatrix} \sim \begin{pmatrix}1 & -3 & -9\\0 & 17 & 53\end{pmatrix} \sim \begin{pmatrix}1 & 0 & \frac{6}{17}\\0 & 1 & \frac{53}{17}\end{pmatrix}$$
   Thus, $x = 6/17$ and $y = 53/17$.
3. $$\begin{pmatrix}1 & -2 & 1 & 3\\2 & -3 & 1 & 8\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 1 & 3\\0 & 1 & -1 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1 & 7\\0 & 1 & -1 & 2\end{pmatrix}$$
   Thus, $x = 7 + z$, $y = 2 + z$, and z is free.
4. $$\begin{pmatrix}2 & -1 & 3\\1 & -1 & 4\\1 & -3 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 4\\0 & 1 & -2\\0 & -2 & -3\end{pmatrix} \sim \begin{pmatrix}1 & -1 & 4\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}$$
   This system is inconsistent, as the last row implies $0 = 1$, so there is no solution.
5. $$\begin{pmatrix}1 & 1 & -1\\2 & -1 & 7\\1 & -2 & 8\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\0 & -3 & 9\\0 & -3 & 9\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1\\0 & 1 & -3\\0 & 0 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 2\\0 & 1 & -3\\0 & 0 & 0\end{pmatrix}$$
   Thus, $x = 2$ and $y = -3$.
6. $$\begin{pmatrix}2 & -4 & 1 & -1\\1 & 2 & -3 & -9\\3 & 2 & 2 & 4\end{pmatrix} \sim \begin{pmatrix}1 & 2 & -3 & -9\\0 & -8 & 7 & 17\\0 & 4 & -11 & -31\end{pmatrix} \sim \begin{pmatrix}1 & 2 & -3 & -9\\0 & 4 & -11 & -31\\0 & 0 & 1 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & 1/2\\0 & 0 & 1 & 3\end{pmatrix}$$
   Thus, $x = -1$, $y = 1/2$ and $z = 3$.
7. $$\begin{pmatrix}5 & -1 & 2 & 6\\1 & 2 & -1 & -1\\3 & 2 & -2 & 1\end{pmatrix} \sim \begin{pmatrix}1 & 2 & -1 & -1\\0 & -11 & 7 & 11\\0 & 1 & -1 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 2 & -1 & -1\\0 & 1 & -1 & -1\\0 & 0 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 0\end{pmatrix}$$
   Thus, $x = 1$, $y = -1$ and $z = 0$.

Exercise 7.3 1.

$$\begin{pmatrix}1 & 3 & 4 & 7\\ 3 & 9 & 7 & 6\end{pmatrix} \sim \begin{pmatrix}1 & 3 & 4 & 7\\ 0 & 0 & 1 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 3 & 0 & -5\\ 0 & 0 & 1 & 3\end{pmatrix}$$

Thus, $x = -5 - 3y$, y is free, and $z = 3$.

2.

$$\begin{pmatrix}1 & 4 & 0 & 7\\ 2 & 7 & 0 & 10\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 0 & 7\\ 0 & 1 & 0 & 4\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -9\\ 0 & 1 & 0 & 4\end{pmatrix}$$

Thus, $x = -9$, $y = 4$ and z is free.

3.

$$\begin{pmatrix}0 & 1 & -6 & 5\\ 1 & -2 & 7 & -6\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -5 & 4\\ 0 & 1 & -6 & 5\end{pmatrix}$$

Thus, $x = 4 + 5z$, $y = 5 + 6z$ and z is free.

4.

$$\begin{pmatrix}1 & -2 & -1 & 3\\ 3 & -6 & -2 & 2\end{pmatrix} \sim \begin{pmatrix}1 & -2 & -1 & 3\\ 0 & 0 & 0 & -7\end{pmatrix}$$

and the system is inconsistent.

5.

$$\begin{pmatrix}3 & -4 & 2 & 0\\ -9 & 12 & -6 & 0\\ -6 & 8 & -4 & 1\end{pmatrix} \sim \begin{pmatrix}3 & -4 & 2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and the system is inconsistent.

6.

$$\begin{pmatrix}1 & -3 & 0 & -1 & 0 & -2\\ 0 & 1 & 0 & 0 & -4 & -1\\ 0 & 0 & 0 & 1 & 9 & 4\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$

thus $x_1 = -1$, $x_2 = -1$, x_3 is free, $x_4 = 4$ and $x_5 = 0$.

7.

$$\begin{pmatrix}1 & 0 & 2 & 6\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & 9\\ 0 & 0 & 0 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0 & -3\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & 9\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Thus $x = -3$, $y = 4$ and $z = 9$.

8.

$$\begin{pmatrix}1 & 2 & -5 & -6 & 0 & -5\\ 0 & 1 & -6 & -3 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 7 & 0 & 0 & -9\\ 0 & 1 & -6 & -3 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

Thus $x_1 = - - 9 - 7x_3$, $x_2 = 2 + 6x_3 + 3x_4$, x_3 and x_4 are free, and $x_5 = 0$.

Exercise 7.4 1. The augmented matrix of the system is

$$\begin{pmatrix}2 & 3 & h\\ 4 & 6 & 7\end{pmatrix} \sim \begin{pmatrix}2 & 3 & h\\ 0 & 0 & 7 - 2h\end{pmatrix},$$

and therefore the system is consistent if $h = 7/2$.

2. The augmented matrix of the system is

$$\begin{pmatrix}1 & -3 & -2\\ 5 & h & -7\end{pmatrix} \sim \begin{pmatrix}1 & -3 & -2\\ 0 & h + 15 & 3\end{pmatrix},$$

and therefore the system is consistent if $h \neq -15$.

Exercise 7.5 1. The augmented matrix of the system is

$$\begin{pmatrix} 1 & h & 2 \\ 4 & 8 & k \end{pmatrix} \sim \begin{pmatrix} 1 & h & 2 \\ 0 & 8-4k & k-8 \end{pmatrix}.$$

If $h \neq 2$, the system has a unique solution. If $h = 2$ but $k \neq 8$, the system has no solution. Finally, if $h = 2$ and $k = 8$, the system has infinite solutions.

2. The augmented matrix of the system is

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & h & k \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & h-9k & k-6 \end{pmatrix}.$$

If $h \neq 9$, the system has a unique solution. If $h = 9$ but $k \neq 6$, the system has no solution. Finally, if $h = 9$ and $k = 6$, the system has infinite solutions.

Exercise 7.6 1. The system is

$$\begin{cases} x_1 + x_2 = 3 \\ 4x_1 + 5x_2 = 8 \end{cases}$$

whose augmented matrix is

$$\begin{pmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{pmatrix},$$

and the solution is $(7, 4)$.

2. The system is

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_1 + 3x_2 = 4 \\ 3x_1 - 2x_2 = 9 \end{cases}$$

whose augmented matrix is

$$\begin{pmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ 3 & -2 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

and the system has no solution.

A.8 Matrix Algebra

Exercise 8.1 The standard matrix of T is

$$\begin{pmatrix} 1 & 4 & 5 \\ 3 & 7 & 4 \end{pmatrix}$$

Exercise 8.2 1. $-2A = \begin{pmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{pmatrix}$

2. $B - 2A = \begin{pmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{pmatrix}$
3. AC can't be computed, since A is a 2×3 matrix and C is 2×2. The dimensions don't match.
4. $CD = \begin{pmatrix} 1 & 13 \\ -7 & -6 \end{pmatrix}$
5. $A + 2B = \begin{pmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{pmatrix}$
6. $3C - E$ can't be computed, since $3C$ is a 2×2 matrix and E is 2×1.
7. $CB = \begin{pmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{pmatrix}$
8. EB can't be computed since E is a 2×1 matrix and B is 2×3.

Exercise 8.3 We have

$$AB = \begin{pmatrix} 23 & -10 + 5k \\ -9 & 15 + k \end{pmatrix} \quad BA = \begin{pmatrix} 23 & 15 \\ 6 - 3k & 15 + k \end{pmatrix},$$

so k must satisfy the two equations

$$\begin{cases} -10 + 5k = 15 \\ 6 - 3k = -9 \end{cases} \implies \begin{cases} 5k = 25 \\ 3k = 15 \end{cases}$$

which has as unique solution $k = 5$.

Exercise 8.4 We have

$$AB = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix} \quad AC = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}.$$

The equality appears because $\det A = 0$ and so there are infinite vectors that satisfy $A\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and infinite vectors that satisfy $A\mathbf{x} = \begin{pmatrix} -7 \\ 14 \end{pmatrix}$.

Exercise 8.5 1. $\begin{pmatrix} = 2 & -3 \\ -5/2 & 4 \end{pmatrix}$

2. $\begin{pmatrix} = -2 & 1 \\ 7/2 & -3/2 \end{pmatrix}$
3. $\begin{pmatrix} = 1 & 1 \\ -7/5 & -8/5 \end{pmatrix}$
4. $\begin{pmatrix} = -2 & 1 \\ -7/4 & 3/4 \end{pmatrix}$

Exercise 8.6 If $\det A \neq 0$, the system $A\mathbf{x} = \mathbf{b}$ has as unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. In these cases,

1. $x = \begin{pmatrix} = 2 & -3 \\ -5/2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
2. Here we need to multiply the second equation by -1, so that the coefficient matrix is the same as in exercise 8.4.1. $x = \begin{pmatrix} = 1 & 1 \\ -7/5 & -8/5 \end{pmatrix} \begin{pmatrix} 9 \\ -11 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

Exercise 8.7 1. $\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -\det A$, which means that exchanging two rows changes the signs of the determinant.
2. $\det \begin{pmatrix} a + kc & b + kd \\ c & d \end{pmatrix} = ad - bc = \det A$, which means that substituting one row by a linear combination of rows from the matrix doesn't change the determinant.
3. $\det \begin{pmatrix} a & b \\ kc & kd \end{pmatrix} = kad - kbc = k \det A$, which means that scaling a row by a number k multiplies the determinant by k too.

Exercise 8.8 The area of the parallelogram $OABC$ is equal to the area of the rectangle with vertices at O and B (with sides $a + b$ and $c + d$) minus the area of the two rectangles with sides b and c (check that the two squares are identical), minus the area of the two identical right triangles with sides a and c (one below the paralellogram and one above it), and minus the area of the two identical right triangles with sides b and d (one to the left of the parallelogram, one to its right). Summing everything:

$$\text{Area} = (a + b)(c + d) - 2bc - 2\frac{1}{2}ac - 2\frac{1}{2}bd = ad - bc.$$

Exercise 8.9 We can do this in two ways.

1. The first is to calculate $A\mathbf{b}_1 = \begin{pmatrix} -21 \\ 12 \end{pmatrix}$ and $A\mathbf{b}_2 = \begin{pmatrix} -27 \\ 16 \end{pmatrix}$ and calculate the are of the paralellogram formed by these two vectors:

$$\begin{vmatrix} -21 & -27 \\ 12 & 16 \end{vmatrix} = -21 \cdot 16 + 12 \cdot 27 = 3 \cdot 4 \cdot (-7 \cdot 4 + 9 \cdot 3) = -12.$$

2. The other one is to calculate the area of S

$$\begin{vmatrix} -2 & -2 \\ 3 & 5 \end{vmatrix} = -10 + 6 = -4,$$

and then calculate the determinant of A, which will give us the expansion of the area of S:

$$\det A = \begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} = 3.$$

So

$$\text{Area of } S(A) = \det A \cdot \text{Area of } S = -12.$$

We have kept the signs in order to make clear that both approaches are identical, but since we are being asked about an area, the answer is 12.

Exercise 8.10 We have

$$AB = \begin{pmatrix} 7 & 40 \\ -5 & 26 \end{pmatrix} \implies \det AB = 18.$$

On the other hand,

$$\det A = 9, \ \det B = 2 \implies \det A \det B = 18 = \det AB.$$

If we think of A and B as the matrices of two linear transformations, the change in area caused by the action of first B and then A (or vice versa) is the same as that caused by the composition AB.

Exercise 8.11 We can write

$$\begin{pmatrix} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{pmatrix},$$

and so $A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

A.9 Eigenvalues and Eigenvectors

Exercise 9.1 1. $\lambda_1 = 2, \lambda_2 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. $\lambda_1 = 0, \lambda_2 = -3, \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
3. $\lambda_1 = +3, \lambda_2 = -3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
4. $\lambda_1 = 2, \lambda_2 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Exercise 9.2 1. The characteristic equation is $\lambda^2-(a+d)\lambda+(ad-bc) = 0$. Since tr $A = a + d$ and $etA = ad - bc$, the identity of the two expressions is immediate.

2. Expanding the product, we obtain $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$, from which we derive the result.
3. We have $\lambda_1 = \text{tr } A - \lambda_2$ and so tr $A\lambda_2 - \lambda_2^2 = \det A$. Solving this quadratic equation yields the result.

Exercise 9.3 1. $\lambda_1 = -1, \lambda_2 = 2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

2. The system $\begin{pmatrix} 1 \\ -3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ has as unique solution $a = 2, b = -1$.
3. Since $A\mathbf{x} = A(a\mathbf{v}_1 + b\mathbf{v}_2) = a\lambda_1\mathbf{v}_1 + b\lambda_2\mathbf{v}_2$ and, in this case, $a = 2, b = -1$, we have

$$A^{20}\mathbf{x} = 2(-1)^{20}\mathbf{v}_1 + 2^{20}\mathbf{v}_2 = \begin{pmatrix} 2 + 2^{20} \\ -3 \cdot 2^{20} \end{pmatrix} = \begin{pmatrix} -1048574 \\ -3145728 \end{pmatrix}.$$

Exercise 9.4 The eigenvalues and eigenvectors of A are $\lambda_1 = -1, \lambda_2 = 1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The vector $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ can be expressed as a linear combination of the two eigenvectors (you need to solve the 2×2 system of equations) as follows: $\mathbf{x} = 2\mathbf{v}_1 + 2\mathbf{v}_2$.

Hence, we have

$$A^{15} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\lambda_1^{15}\mathbf{v}_1 + 2\lambda_2^{15}\mathbf{v}_2 = -2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

Exercise 9.5 1. The terms in the sequence are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

2. Since $A_{n+1} = A_n + Y_n$, we can substitute $Y_n = A_{n-1}$ to obtain $A_{n+1} = A_n + A_{n-1}$ or, making $k = n - 1$, $A_{k+2} = A_{k+1} + A_k$.
3. The eigenvalues are $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$. The first number is called the **golden number**, or ϕ. The second number is actually equal to $1 - \phi$ or $-\phi^{-1}$. The corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ \phi \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -\phi \\ 1 \end{pmatrix}$.

4. Since $\begin{pmatrix} Y_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ \phi \end{pmatrix} + b \begin{pmatrix} -\phi \\ 1 \end{pmatrix}$, we can solve the system to obtain $a = \frac{1}{1+\phi^2}$ and $b = -\frac{\phi}{1+\phi^2}$ or, operating, $a = 2/(5+\sqrt{5})$, $b = -1/(1+\sqrt{5})$. In any case,
$$\begin{pmatrix} Y_n \\ A_n \end{pmatrix} = \frac{\phi^n}{1+\phi^2} \begin{pmatrix} 1 \\ \phi \end{pmatrix} + \frac{(-1)^n \phi^{-n-1}}{1+\phi^2} \begin{pmatrix} -\phi \\ 1 \end{pmatrix}.$$
5. The second eigenvalue $1 - \sqrt{5}/2$ is smaller than 1 in absolute value: $|\lambda_2| < 1$. This means that, raised to a high power n, it becomes more and more small and closer to zero. For instance, $\lambda_2^{10} \approx 0.008$ and $\lambda_2^{20} \approx 0.00006$, whereas $\lambda_1^{20} \approx 15,127$. This means that, for large n, we can actually disregard the term with λ_2^n and therefore
$$\begin{pmatrix} Y_n \\ A_n \end{pmatrix} \approx \frac{\phi^n}{1+\phi^2} \begin{pmatrix} 1 \\ \phi \end{pmatrix}.$$
In particular, this means that the fraction $A_n/Y_n \approx \phi$ when $n \to \infty$. In terms of the Fibonacci sequence, this means that the fractions between two consecutive Fibonacci numbers approaches ϕ as n becomes large.

Exercise 9.6 The eigenvalues of the Leslie matrix are $\lambda_1 = -0.1$, $\lambda_2 = 1.5$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -0.3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. As in the previous exercises, we have
$$\begin{pmatrix} I_t \\ M_t \end{pmatrix} = a(-0.1)^t \begin{pmatrix} 1 \\ -0.3 \end{pmatrix} + b(1.5)^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \approx b(1.5)^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
Now, this means that the fraction $I_t/M_t \approx 2$ as $t \to \infty$, and since the eigenvector is being multiplied by a factor $(1.5)^t$, the total poppulation will keep growing without end.

Exercise 9.7 1. The eigenvalues of the transition matrix are $\lambda_1 = 1.02$, $\lambda_2 = 0.58$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 10 \\ 13 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

2. As in the previous exercises, we have $\mathbf{x}_k = a(1.02)^k \begin{pmatrix} 10 \\ 13 \end{pmatrix} + b(0.58)^k \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.
3. As before, the term $(0.58)^k \to 0$ as $k \to \infty$, so $\mathbf{x}_k \approx a(1.02)^k \begin{pmatrix} 10 \\ 13 \end{pmatrix}$ as k becomes large. This means that the population will keep grwoing (albeit very slowly) with a stable fraction of owls and rats.

Exercise 9.8 1. $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$.

2. $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$.

Exercise 9.9 1. $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$, $A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}$ and so forth.

2. For part 1 we have
$$A^{10} = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -3 \cdot 2^{10} + 4 & 3 \cdot 2^{12} - 12 \\ -2^{10} + 1 & 2^{12} - 3 \end{pmatrix},$$

and for part 2:

$$A^{10} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} a^{10} & 0 \\ 3(a^{10} - b^{10}) & b^{10} \end{pmatrix}.$$

A.10 Complex Numbers

Exercise 10.1 We multiply each side of the equation by $x - i5$, and perform complex numbers multiplication:

$$43+iy = (4+i3)(x-i5) \iff 43+iy = 4x-i20+i3x+15 \iff 43+iy = (4x+15)+(3x-20)i$$

Because the real and imaginary parts of both sides must be equal, we have the linear system

$$\begin{aligned} 4x \quad &= 28 \\ 3x - y &= 20 \end{aligned}$$

which has as unique solution $x = 7$, $y = 1$.

Exercise 10.2

(1) $(2-i)^3 = (2-i)(2-i)(2-i) = (3-4i)(2-i) = 6-3i-8i-4 = 2-11i$

(2) $i^{13} = i^{12+1} = i(i^4)^3 = i$

(3) $\dfrac{1}{i} = \dfrac{i}{i^2} = -i$

(4) $\dfrac{1}{1+2i} = \dfrac{1-2i}{(1+2i)(1-2i)} = \dfrac{1-2i}{5} = \dfrac{1}{5} - \dfrac{2}{5}i$

(5) $\dfrac{1+i}{i-1} = \dfrac{(1+i)(-1-i)}{(-1+i)(-1-i)} = \dfrac{-2i}{2} = -i$

(6) $i + i^2 + i^3 + i^4 = i + (-1) + (-1)i + (-1)^2 = 0$

Exercise 10.3

$$\begin{aligned} z &= \left(\frac{a+bi}{a-bi}\right)^2 + \left(\frac{a-bi}{a+bi}\right)^2 = \left(\frac{(a+bi)^2}{a^2+b^2}\right)^2 + \left(\frac{(a-bi)^2}{a^2+b^2}\right)^2 \\ &= \frac{(a+bi)^4 + (a-bi)^4}{(a^2+b^2)^2} = 2\frac{a^4+b^4-6a^2b^2}{(a^2+b^2)^2}. \end{aligned}$$

As we can see, z is a real number, and therefore $\bar{z} = z$.

Exercise 10.4

(1) $w = z + 3i = (\mathrm{Re}\, z) + (\mathrm{Im}\, z + 3)i \implies \mathrm{Re}\, w = \mathrm{Re}\, z,\ \mathrm{Im}\, w = \mathrm{Im}\, z + 3$

(2) $w = iz = (-\mathrm{Im}\, z) + (\mathrm{Re}\, z)i \implies \mathrm{Re}\, w = -\mathrm{Im}\, z,\ \mathrm{Im}\, w = \mathrm{Re}\, z$

(3) $w = (1+z)(\bar{z}+1) = (\bar{z} + 1 + |z|^2 + z) = 1 + |z|^2 + 2\mathrm{Re}\, z$

$\implies \mathrm{Re}\, w = 1 + |z|^2 + 2\mathrm{Re}\, z,\ \mathrm{Im}\, w = 0$

Exercise 10.5

$$(1)\ \ |-i| = \sqrt{(-1)^2} = 1$$
$$(2)\ \ |1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$(3)\ \ |1-i| = |1+i| = \sqrt{2}$$
$$(4)\ \ |(1+i)^2| = |1+i|^2 = 2$$
$$(5)\ \ \left|\frac{1}{1+i}\right| = \frac{1}{|1+i|} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$(6)\ \ \left|\frac{1}{(1-i)^2}\right| = \frac{1}{|1-i|^2} = \frac{1}{2}$$
$$(7)\ \ |1-\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = 2$$

Exercise 10.6

$$(1)\ \ \mathrm{Arg}\,1+i = \tan^{-1}\left(\frac{\mathrm{Im}\,1+i}{\mathrm{Re}\,1+i}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$
$$(2)\ \ \mathrm{Arg}\,(1+i)^{-1} = \mathrm{Arg}\,1 - \mathrm{Arg}\,1+i = -\frac{\pi}{4}$$
$$(3)\ \ \mathrm{Arg}\,(1+i)^2 = 2\mathrm{Arg}\,1+i = \frac{\pi}{2}$$
$$(4)\ \ \mathrm{Arg}\,(1+i)^3 = 3\mathrm{Arg}\,1+i = \frac{3\pi}{4}$$

Exercise 10.7

$$(1)\ \ e^{\frac{\pi}{4}i} - e^{-\frac{\pi}{4}i} = \Big(\cos(\pi/4) + i\sin(\pi/4)\Big) - \Big(\cos(-\pi/4) + i\sin(-\pi/4)\Big) =$$
$$= \Big(\cos(\pi/4) - \cos(-\pi/4)\Big) + i\Big(\sin(\pi/4) - \sin(-\pi/4)\Big) = 2\sin(\pi/4)i = \sqrt{2}i$$
$$(2)\ \ \frac{1-e^{\frac{\pi}{2}i}}{1+e^{\frac{\pi}{2}i}} = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{-2i}{2} = -i$$
$$(3)\ \ e^{\pi i}(1-e^{-\frac{\pi}{3}i}) = -(1-(\cos(-\pi/3)+i\sin(-\pi/3))) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Exercise 10.8

$$(1)\ \ |-i| = 1,\ \mathrm{Arg}\,-i = -\frac{\pi}{2} \implies -i = e^{-i\pi/2}$$
$$(2)\ \ |1+i| = \sqrt{2},\ \mathrm{Arg}\,1+i = \frac{\pi}{4} \implies 1+i = \sqrt{2}e^{i\pi/4}$$
$$(3)\ \ |1-i| = \sqrt{2},\ \mathrm{Arg}\,1-i = -\frac{\pi}{4} \implies 1-i = \sqrt{2}e^{-i\pi/4}$$
$$(4)\ \ |(1+i)^2| = 2,\ \mathrm{Arg}\,(1+i)^2 = \frac{\pi}{2} \implies (1+i)^2 = 2e^{i\pi/2}$$
$$(5)\ \ \left|\frac{1}{1+i}\right| = \frac{\sqrt{2}}{2}, \mathrm{Arg}\,(1+i)^{-1} = -\frac{\pi}{4} \implies \frac{1}{1+i} = \frac{\sqrt{2}}{2}e^{-i\pi/4}$$
$$(6)\ \ \left|\frac{1}{(1-i)^2}\right| = \frac{1}{2}, \mathrm{Arg}\,(1-i)^{-2} = \frac{\pi}{2} \implies \frac{1}{(1-i)^2} = \frac{1}{2}e^{i\pi/2}$$
$$(7)\ \ |1-\sqrt{3}i| = 2,\ \mathrm{Arg}\,1-\sqrt{3}i = \tan^{-1}(-\sqrt{3}) = -\frac{pi}{3} \implies 1-\sqrt{3}i = 2e^{-i\pi/3}$$

Exercise 10.9 1.

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= e^{i3\theta} = (e^{i\theta})^3\\ &= (\cos\theta + i\sin\theta)^3\\ &= (\varrho s^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta),\end{aligned}$$

2. Using the hint:

$$\begin{aligned}2^4\cos^4\theta &= (e^{i\theta} + e^{-i\theta})^4\\ &= e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}\\ &= 2\cos 4\theta + 8\cos 2\theta + 6.\end{aligned}$$

Exercise 10.10 The relationship between Fourier series and complex numbers comes from Euler's formula. Since

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta,$$

we can express cosine and sine terms in terms of complex exponentials:

$$\begin{aligned}\cos\left(\frac{n\pi x}{L}\right) &= \frac{e^{in\frac{\pi x}{L}} + e^{-in\frac{\pi x}{L}}}{2},\\ \sin\left(\frac{n\pi x}{L}\right) &= \frac{e^{in\frac{\pi x}{L}} - e^{-in\frac{\pi x}{L}}}{2i}.\end{aligned}$$

These equalities suggest that we could have chosen to expand $f(x)$ in terms of e^{inx}:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi x}{L}},$$

The coefficients in the complex formulation c_n relate to a_n and b_n as follows:

$$\begin{aligned}c_n &= \frac{a_n - ib_n}{2} \quad \text{for } n > 0,\\ c_{-n} &= \frac{a_n + ib_n}{2}.\end{aligned}$$

or, in general:

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\frac{\pi x}{L}}\,dx.$$

Exercise 10.11 1. The eigenvalues are $\sqrt{3} \pm i$. The action of the matrix is equivalent to multiplying by the complex number $z = \sqrt{3} + i$, which in turn can be decomposed into a pure rotation of angle $\pi/6$, $e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ and a scaling by a factor 2.

2. The eigenvalues are $1 \pm 2i$. But the matrix is not in the form $\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$, so
   the action of the original matrix is a rotation composed with a different transformation that scales two lines differently.
3. The eigenvalues are $5/2 \pm i\sqrt{23}/2$. As in the previous example, this matrix is not in the form $\begin{pmatrix}5/2 & \sqrt{23}/2\\ -\sqrt{23}/2 & 5/2\end{pmatrix}$. We can also see that

$$\begin{pmatrix}3 & 3\\ -2 & 2\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},$$

where the last matrix has eigenvalues $\lambda = 1 \pm i$. So we could understand this transformation as a rotation of angle $\pi/4$ times a linear transformation that triples the x axis and doubles the y axis. This descomposition is not unique. In Linear Algebra textbooks you can find a standard decomposition of matrices with complex eigenvalues, but we will not see it here.

4. The eigenvalues are $\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i$, which is a pure rotation of angle $\pi/4$.

Exercise 10.12

$$(1)\ \ z = (-i)^{1/2} \iff z^2 = -i \iff z^2 = e^{-i\pi/2+2\pi k}, k \in \mathbb{Z} \iff z = e^{-i\pi/4+\pi k}$$
$$\implies z_1 = e^{-i\pi/4},\ \ z_2 = e^{i3\pi/4}$$

$$(2)\ \ z = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{1/2} \iff z^2 = e^{-2\pi/3+2\pi k}, k \in \mathbb{Z} \iff z = e^{-\pi/3+\pi k}$$
$$\implies z_1 = e^{-\pi/3+\pi k},\ \ z_2 = e^{2\pi/3+\pi k}$$

$$(3)\ \ z = (-1)^{1/4} \iff z^4 = e^{i\pi+2\pi k} k \in \mathbb{Z} \iff z = e^{i\pi/4+k\pi/2}$$
$$\implies z_1 = e^{-i3\pi/4},\ \ z_2 = e^{-i\pi/4},\ \ z_3 = e^{i\pi/4},\ \ z_4 = e^{i3\pi/4}$$

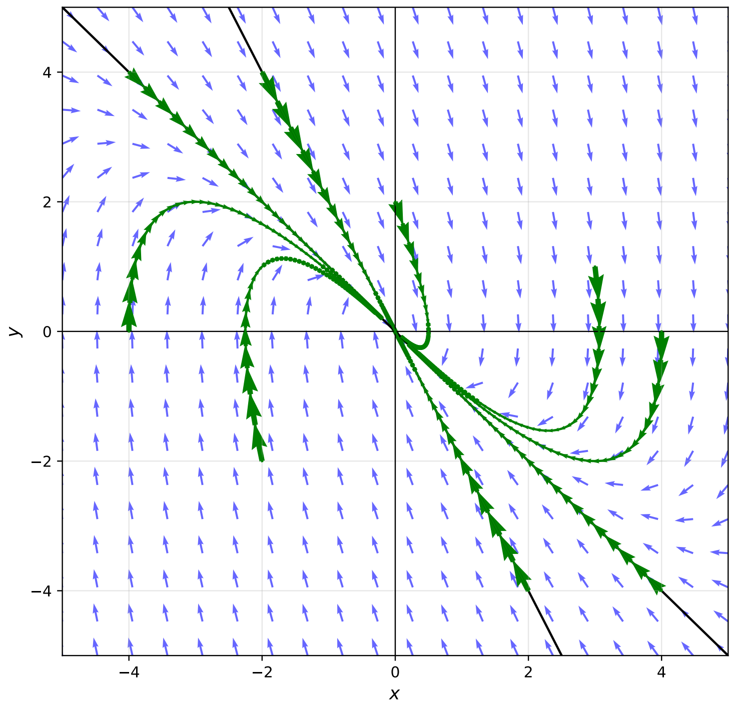
$$(4)\ \ z = 1^{1/6} \iff z^6 = e^{i2\pi k}, k \in \mathbb{Z} \iff z = e^{i\pi k/3}$$
$$\implies z_1 = e^{-i2\pi/3},\ \ z_2 = e^{-i\pi/3},\ \ z_3 = 1,\ \ z_4 = e^{i\pi/3},\ \ z_5 = e^{i2\pi/3},\ \ z_6 = -1$$

A.11 Systems of Linear Differential Equations

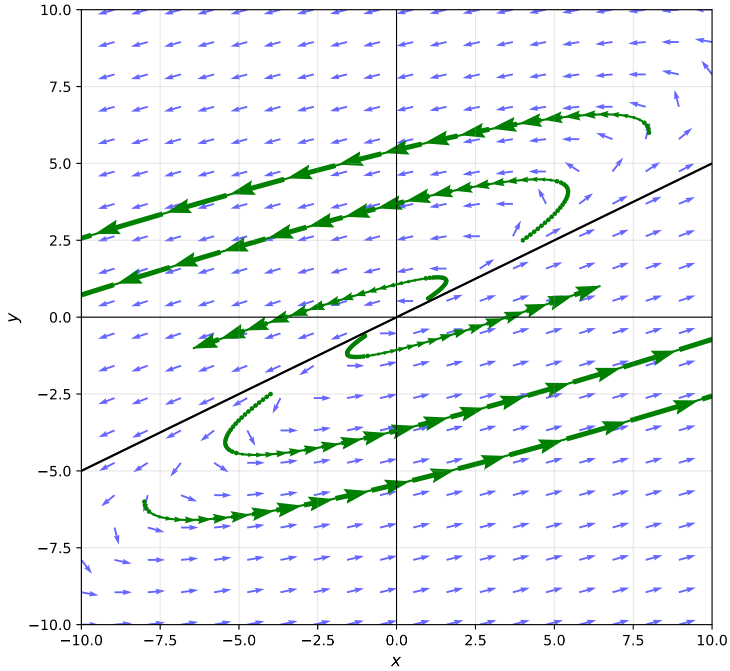
Exercise 11.1 1. The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The fixed point is a stable node.



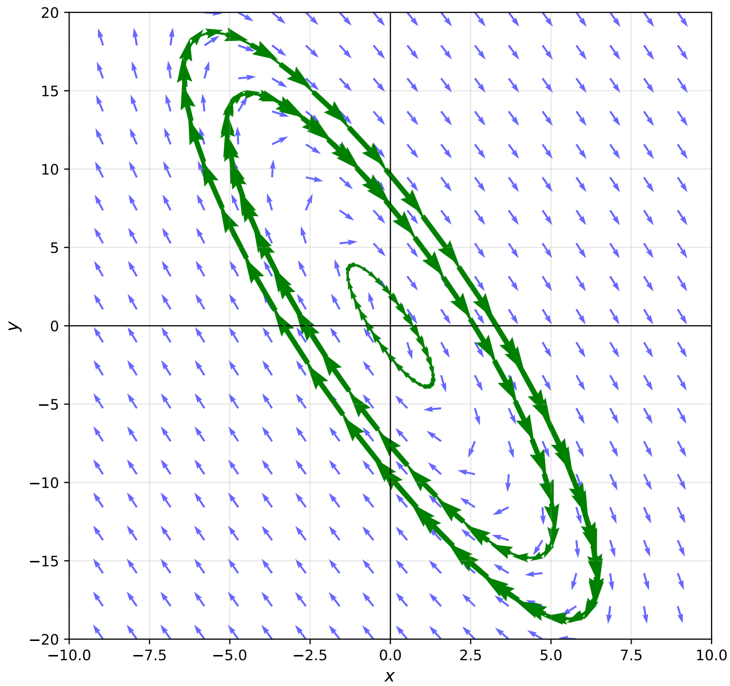
2. The general solution falls outside the scope of this course, as the matrix has only one eigenvalue $\lambda = 1$ and one eigenvector $\mathbf{v} = (2, 1)$. The fixed point is an unstable degenerate node, with only one eigenvector. For more on degenerate nodes, see Exercise 11.7.



3. The general solution is

$$\mathbf{x}(t) = c_1 e^{3it} \begin{pmatrix} -5-3i \\ 17 \end{pmatrix} + c_2 e^{-3it} \begin{pmatrix} -5+3i \\ 17 \end{pmatrix}.$$

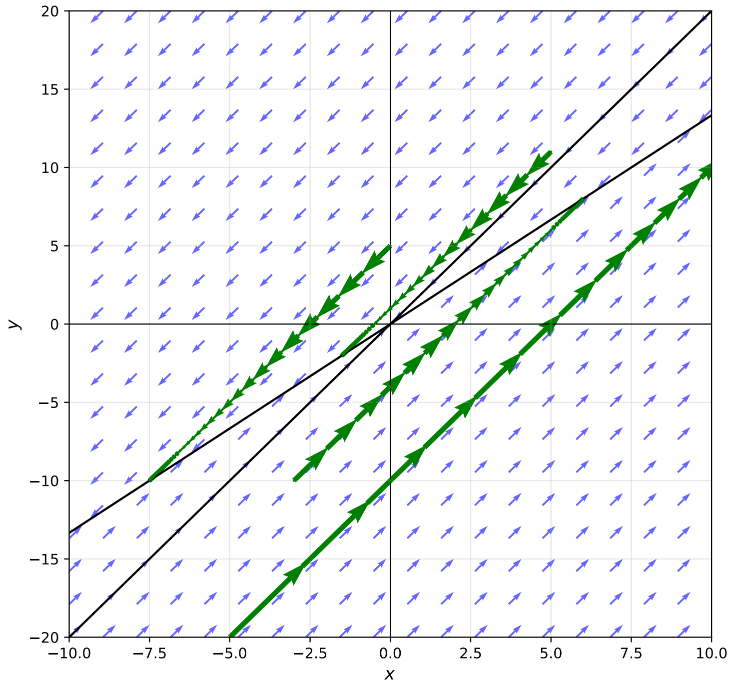
The fixed point is a center.



4. The general solution is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

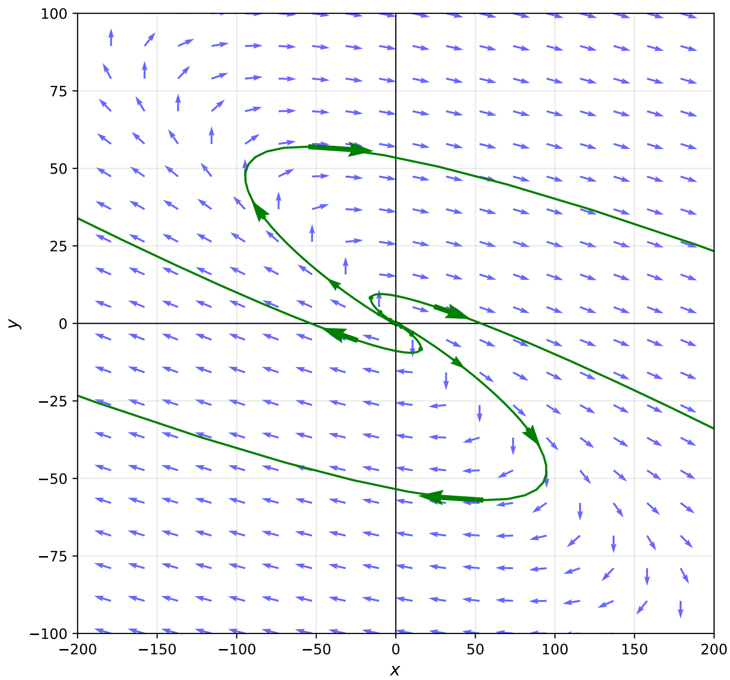
There are infinite fixed points, all of them neutrally stable.



5. The general solution is

$$\mathbf{x}(t) = c_1 e^{(2+i)t} \begin{pmatrix} -3 - i \\ 1 \end{pmatrix} + c_2 e^{(2-i)t} \begin{pmatrix} -3 + i \\ 1 \end{pmatrix}.$$

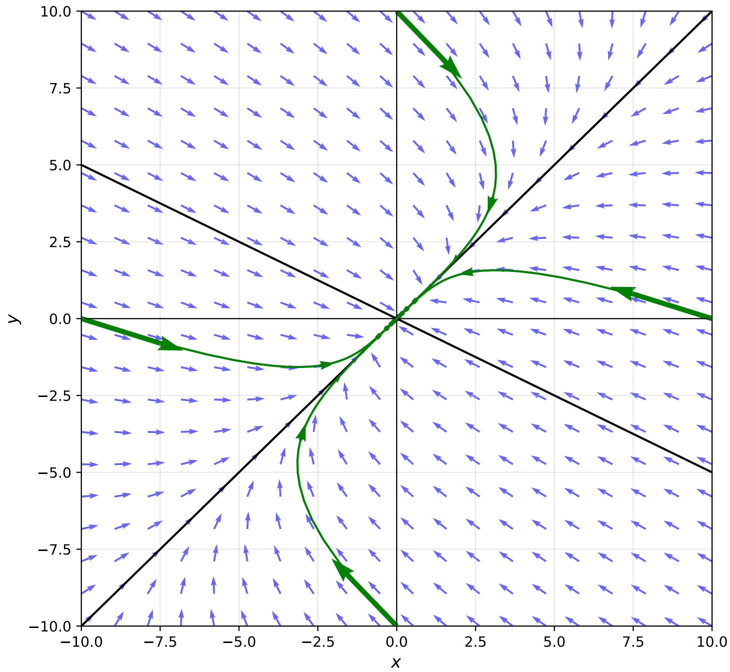
The fixed point is an unstable spiral.



6. The general solution is

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

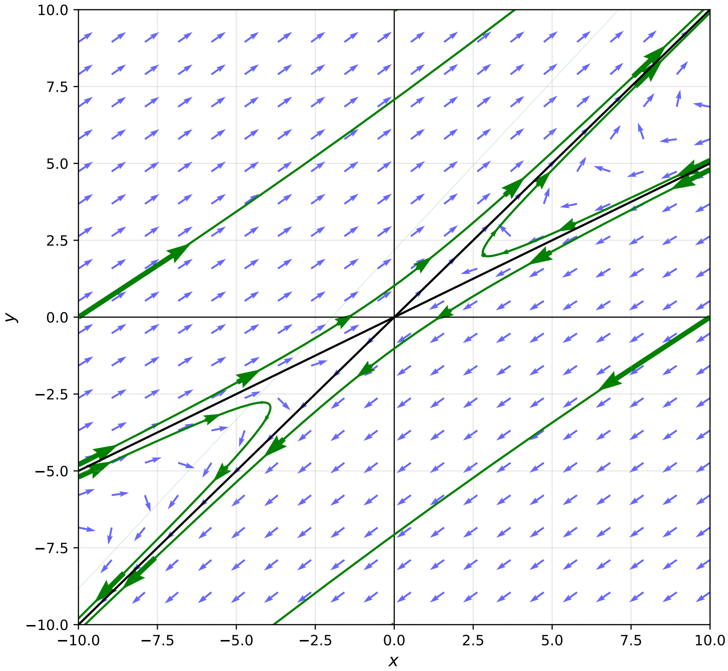
The fixed point is a stable node.



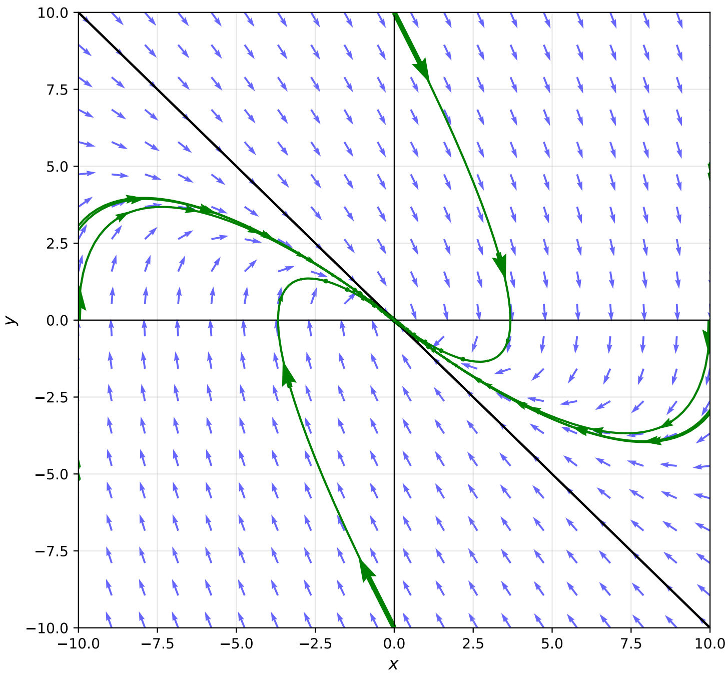
7. The general solution is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The fixed point is a saddle node.

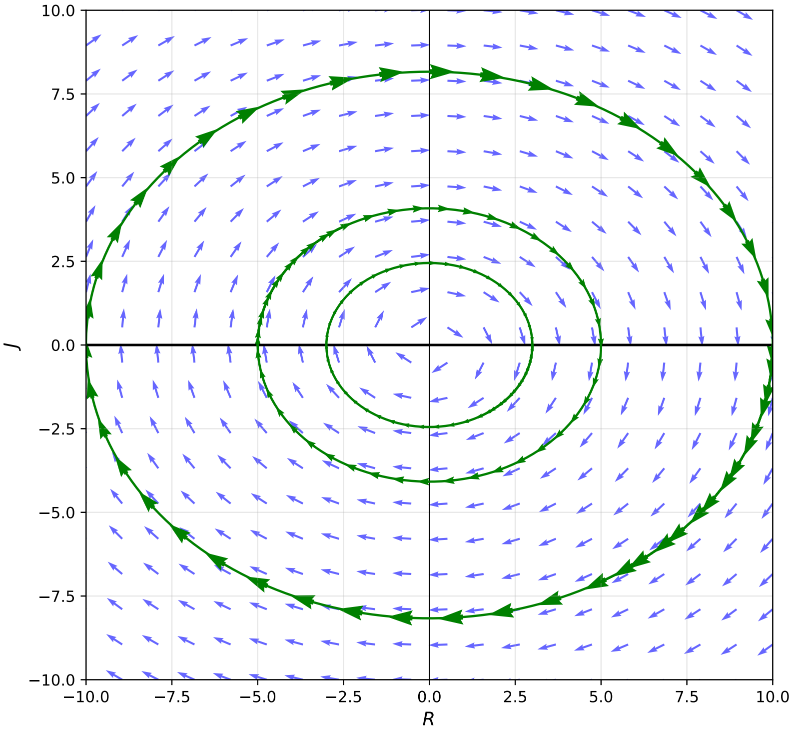


8. The general solution falls outside of the scope of this course: the matrix has only one eigenvalue $\lambda = -1$ with one eigenvector $\mathbf{v} = (-1, 1)$. The fixed point is a stable degenerate node. For more on degenerate nodes, see Exercise 11.7.

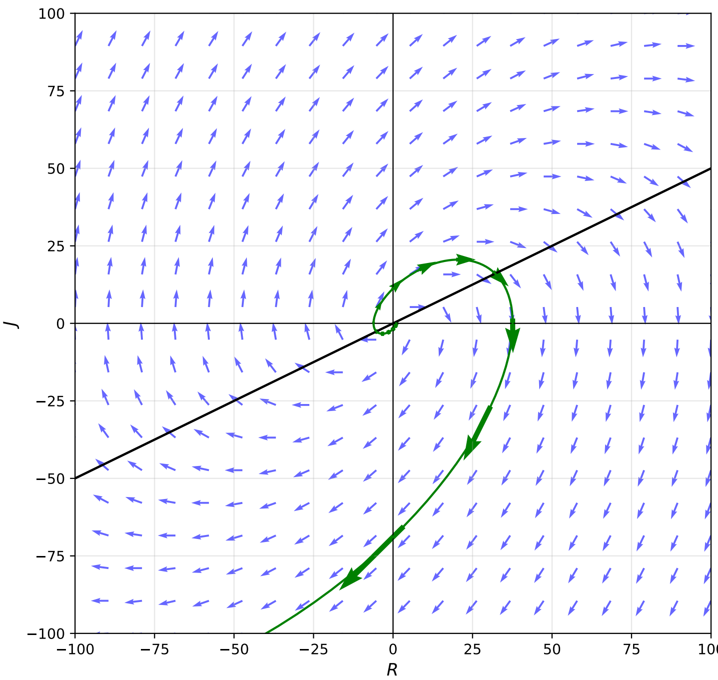


Exercise 11.2 There are two imaginary eigenvalues, $\lambda_{1,2} = \pm i\sqrt{ab}$ which makes the origin a center. The trajectories will oscillate around the origin: if Romeo startes loving Juliet (in the first quadrant), she starts hating him, and the trajectories go downward. Then, Romeo will become more indifferent towards Juliet, and trajectories will turn to the left. At some

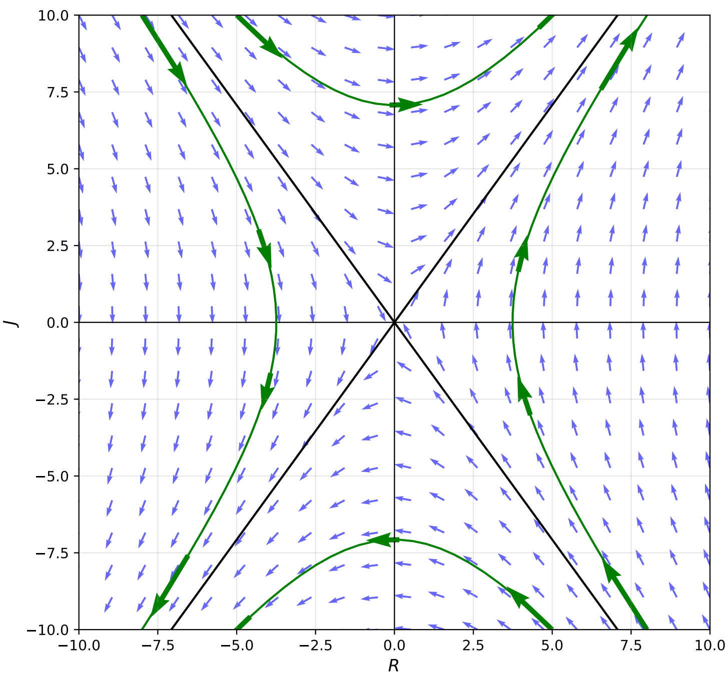
point, Romeo's indifference will trigger Juliet's love, making the trajectory turn upward. Finally, As Juliet now loves him, Romeo will start loving her back, turning the trajectory to the right.



Exercise 11.3
1. The eigenvalues are $\lambda_{1,2} = -1 \pm b$, with corresponding eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$, so the behavior of the system will depend on whether $a > b$ (stable node), $a = b$ (infinite line of stable nodes) or $a < b$ (saddle node).
2. The eigenvalues are $\lambda_{1,2} = 1/2 \pm \sqrt{3}/2i$, so the fixed point is an unstable spiral.



3. There are two different real eigenvalues $\pm\sqrt{bc}$ and so the fixed point is a saddle point. Depending on the initial conditions, Romeo and Juliet's love for each other will either go to infinity, or to minusn infinity (if the initial love is below a given threshold).



Exercise 11.4 There are two eigenvalues, $-k_1$ and 0, and the corresponding eigenvectors are $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (0, 1)$. So the general solution is

$$\mathbf{x}(t) = c_1 e^{-k_1 t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1 e^{-k_1 t}\\ c_2 - c_1 e^{-k_1 t}\end{pmatrix}.$$

We could have also solved this system directly, as $x_1(t) = c_1 e^{-k_1 t}$ from the first equation, and then we can integrate $\dot{x}_2(t) = c_1 k_1 e^{-k_1 t}$ to $x_2(t) = c_2 - c_1 e^{-k_1 t}$.

Since $x_1(0) = K, x_2(0) = 0$, then we can substitute in the general solution to obtain $c_1 = K, c_2 = -K$.

Exercise 11.5 Writing the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases}$$

where $\omega^2 = k/m$, we see that there are two complex eigenvalues, $\pm i\omega$, with corresponding eigenvectors $\mathbf{v}_{1,2} = (\mp i, \omega)$. We can write the solution in terms of the complex eigenvalues and operate, or we can remember that, in this case,

$$\mathbf{x}(t) = c_1 \text{Re}\,(e^{i\omega t}\mathbf{v}_1) + c_2 \text{Im}\,(e^{i\omega t}\mathbf{v}_1 = c_1 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix} + c_2 \begin{pmatrix} -\cos \omega t \\ \omega \sin \omega t \end{pmatrix}.$$

Since $x(0) = 0, v(0) = 1$, we have $c_1 = 1/\omega, c_2 = 0$ and therefore

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \frac{1}{\omega} \sin \omega t \\ \\ cos \omega t \end{pmatrix} \implies x(t) = \frac{1}{\omega} \sin \omega t.$$

Exercise 11.6

1. Since $\dot{x}_1 + \dot{x}_2 = 0$ it must be that $x_1(t) + x_2(t) = A$, a constant. This makes sense, since it measures the total area of the forest, which remains constant.
2. Since $x_1(0)+x_2(0) = 20$, then it is always the case that $x_1(t)+x_2(t) = 20$, and so $x_2(t) = 20 - x_1(t)$, which substituted in the first equation yields $\dot{x}_1 = 2 - 0.3x_1$.
3. The eigenvalues of the system are $\lambda_1 = 0$ and $\lambda_2 = -0.3$, with corresponding eigenvectors $\mathbf{v}_1 = (1, 2)$ and $\mathbf{v}_2 = (-1, 1)$. The general solution is then

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-0.3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

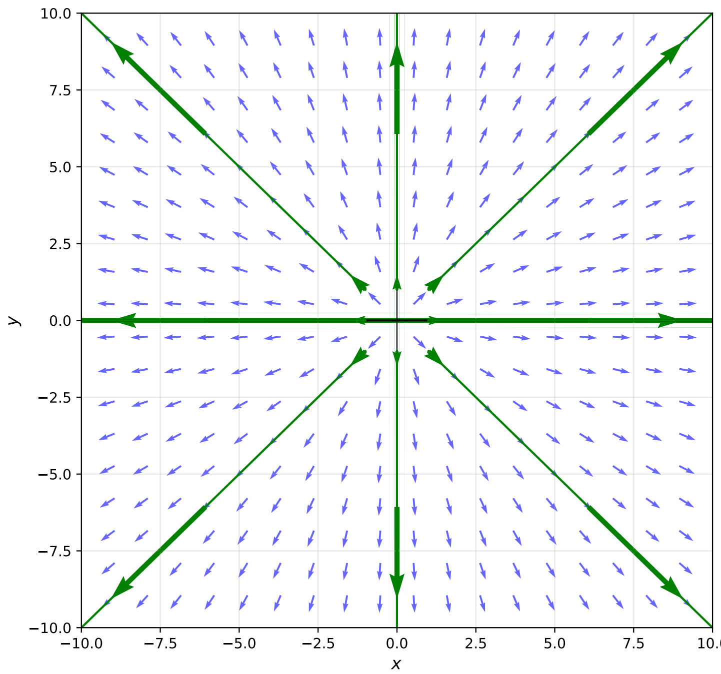
Since $x_1(0) = 2, x_2(0) = 18$, we have $c_1 = 20/3, c_2 = 14/3$ and so

$$\mathbf{x}(t) = \frac{1}{3} \begin{pmatrix} 20 - 14e^{-0.3t} \\ 40 + 14e^{-0.3t} \end{pmatrix},$$

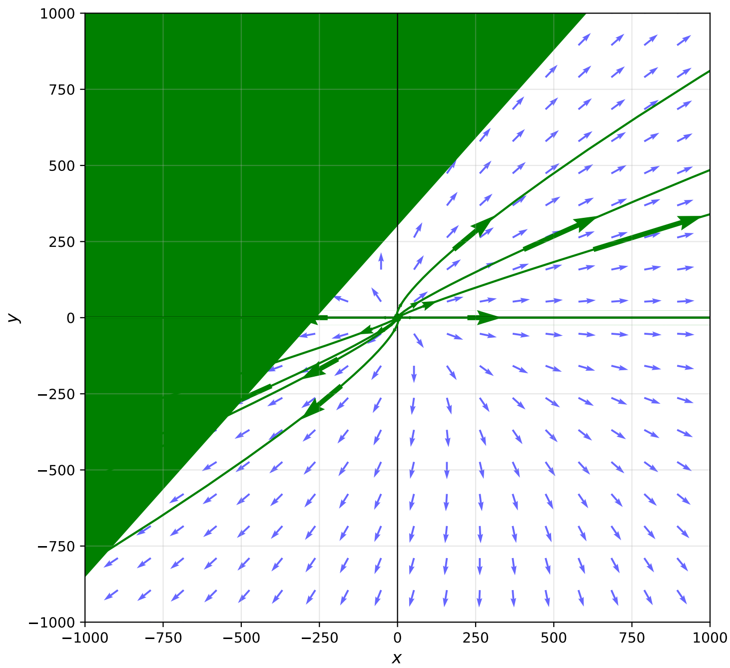
and as $t \longrightarrow \infty$ the forest reaches a stable equilibrium $(20/3, 40/3)$.

Exercise 11.7

1. Here all vectors in the plane are eigenvectors, and the trajectories are paralell to the initial conditions for all time.



2. This is a case of a degenerate unstable node. There is only one eigenvector $(1, 0)$.



From Strogatz:

> A good way to think about the degenerate node is to imagine that it has been created by deforming an ordinary node. The ordinary node has two independent eigendirections; all trajectories are parallel to the slow eigendirection as $t \longrightarrow \infty$, and to the fast eigendirection as $t \longrightarrow -\infty$.
>
> Now suppose we start changing the parameters of the system in such a way that the two eigendirections

are scissored together. Then some of the trajectories will get squashed in the collapsing region between the two eigendirections, while the surviving trajectories get pulled around to form the degenerate node. . Another way to get intuition about this case is to realize that the degenerate node is on the borderline between a spiral and a node. The trajectories are trying to wind around in a spiral, but they don't quite make it.

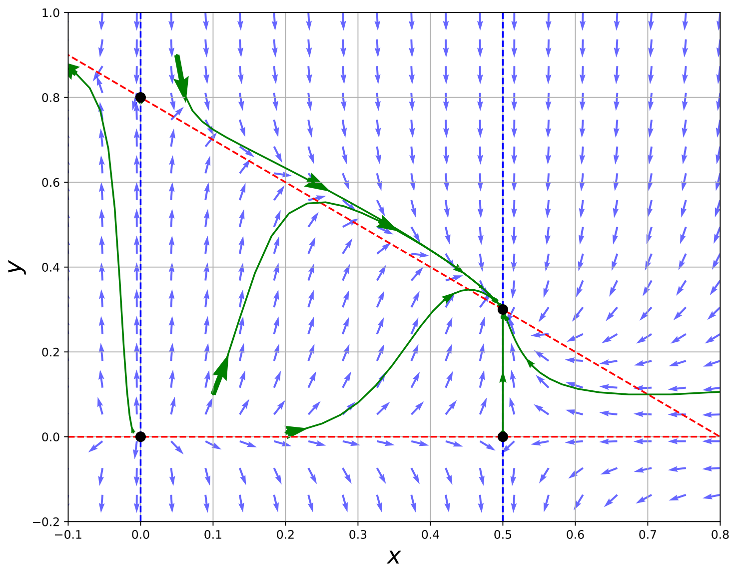
A.12 Systems of Nonlinear Differential Equations

Exercise 12.1 1. The fixed points are $(0,0)$, $(0,4/5)$, $(1/2,0)$, $(1/2,3/10)$. The Jacobian matrix is

$$D\mathbf{f}(x,y) = \begin{pmatrix} 1-4x & 0 \\ -5y & 4-5x-10y \end{pmatrix}.$$

Now, for each fixed point:

- $D\mathbf{f}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, so $\lambda_1 = 1, \lambda_2 = 4$ and the origin is an unstable node.
- $D\mathbf{f}(0,4/5) = \begin{pmatrix} 1 & 0 \\ -4 & -4 \end{pmatrix}$, so $\lambda_1 = 1, \lambda_2 = -4$ and the equilibrium is a saddle point.
- $D\mathbf{f}(1/2,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3/2 \end{pmatrix}$, so $\lambda_1 = -2, \lambda_2 = 3/2$ and the equilibrium is a saddle point.
- $D\mathbf{f}(1/2,3/10) = \begin{pmatrix} -2 & 0 \\ -15/10 & -3/2 \end{pmatrix}$, so $\lambda_1 = -2, \lambda_2 = -3/2$ and the equilibrium is a stable node.



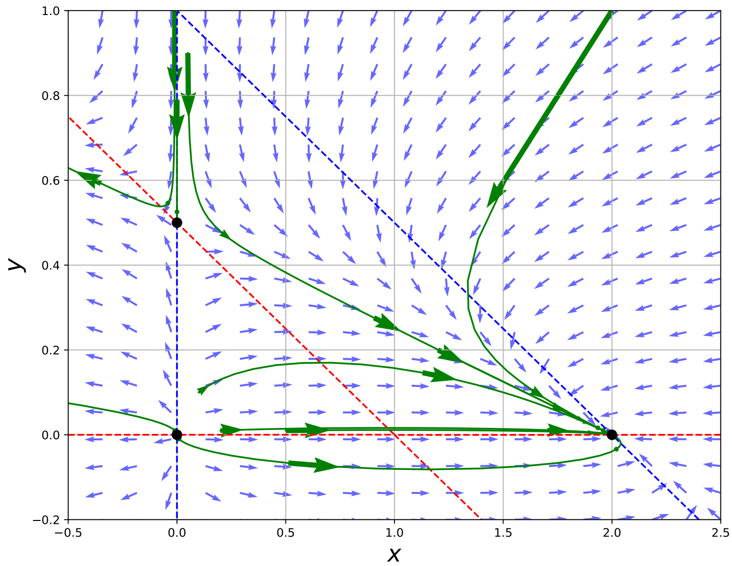
2. The fixed points are $(0,0), (0,1/2), and (2,0)$. The Jacobian matrix is

$$D\mathbf{f}(x,y) = \begin{pmatrix} 2-2x-2y & -2x \\ -y & 1-4y-x \end{pmatrix}.$$

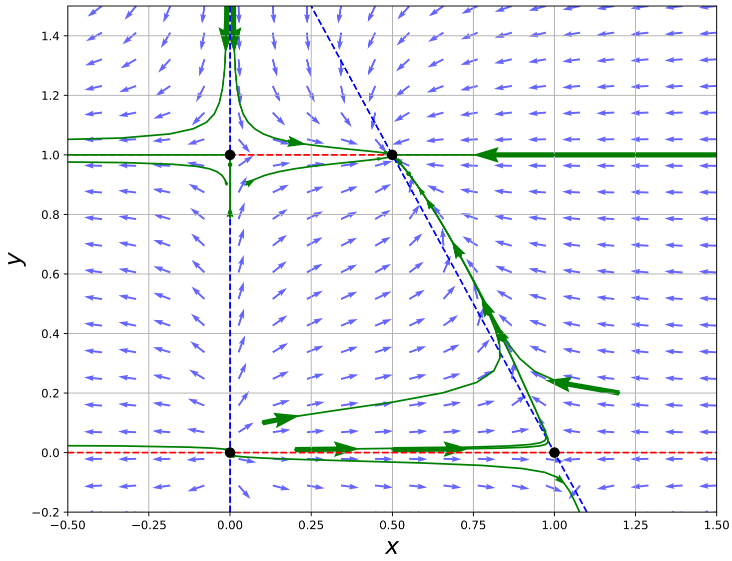
Now, for each fixed point:

- $D\mathbf{f}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, so $\lambda_1 = 2, \lambda_2 = 1$ and the origin is an unstable node.
- $D\mathbf{f}(0,1/2) = \begin{pmatrix} 1 & 0 \\ -1/2 & -1 \end{pmatrix}$, so $\lambda_1 = 1, \lambda_2 = -1$ and the equilibrium is a saddle point.

- $D\mathbf{f}(2,0) = \begin{pmatrix} -2 & -4 \\ 0 & -1 \end{pmatrix}$, so $\lambda_1 = -2, \lambda_2 = -1$ and the equilibrium is a stable node.



3. The fixed points are $(0,0), (0,1), (1,0), (1/2,1)$. The Jacobian matrix is
$$D\mathbf{f}(x,y) = \begin{pmatrix} 4-8x-2y & -2x \\ 0 & 1-2y \end{pmatrix}.$$
Now, for each fixed point:
   - $D\mathbf{f}(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$, so $\lambda_1 = 1, \lambda_2 = 4$ and the origin is an unstable node.
   - $D\mathbf{f}(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, so $\lambda_1 = 2, \lambda_2 = -1$ and the equilibrium is a saddle point.
   - $D\mathbf{f}(1,0) = \begin{pmatrix} -4 & -2 \\ 0 & 1 \end{pmatrix}$, so $\lambda_1 = -4, \lambda_2 = 1$ and the equilibrium is a saddle point.
   - $D\mathbf{f}(1/2,1) = \begin{pmatrix} -2 & -1 \\ 0 & -1 \end{pmatrix}$, so $\lambda_1 = -2, \lambda_2 = -1$ and the equilibrium is a stable node.

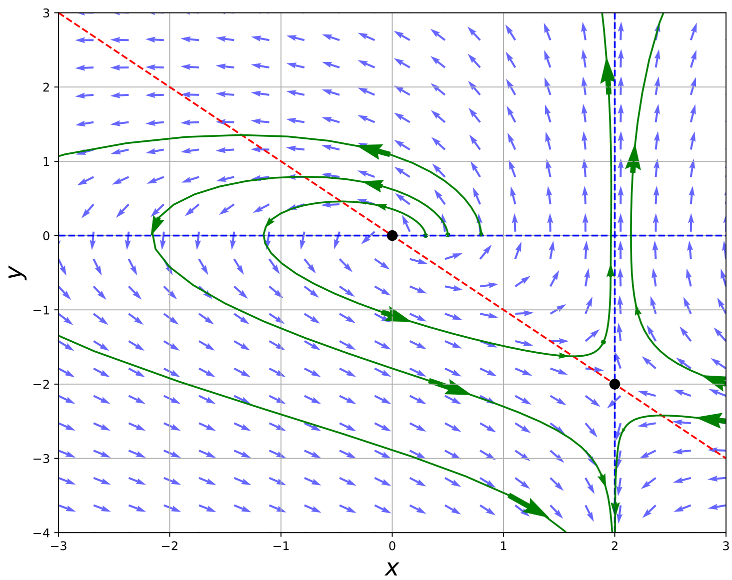


4. The fixed points are $(0,0), and (2,-2)$. The Jacobian matrix is

$$D\mathbf{f}(x,y) = \begin{pmatrix} y & x-2 \\ 1 & 1 \end{pmatrix}.$$

Now, for each fixed point:

- $D\mathbf{f}(0,0) = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$, so $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$, and the origin is an unstable spiral.
- $D\mathbf{f}(2,-2) = \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}$, so $\lambda_1 = -2, \lambda_2 = 1$ and the equilibrium is a saddle point.

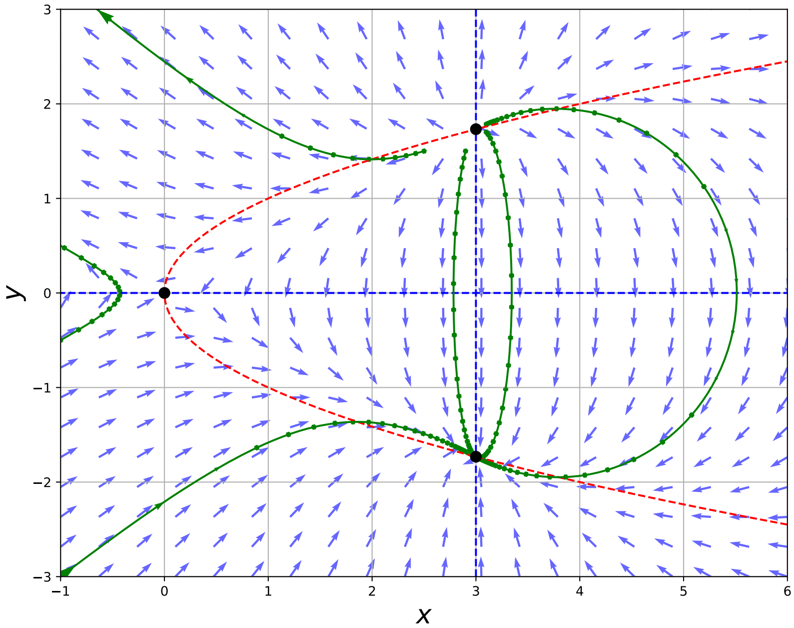


Exercise 12.2 There are three fixed points: $(0,0)$ and $(a, \pm\sqrt{a})$. The Jacobian matrix is

$$D\mathbf{f}(x,y) = \begin{pmatrix} y & x-a \\ -1 & 2y \end{pmatrix}.$$

Now, for each fixed point:

- $D\mathbf{f}(0,0) = \begin{pmatrix} 0 & -a \\ -1 & 0 \end{pmatrix}$, so $\lambda_1 = \sqrt{a}$ and $\lambda_2 = -\sqrt{a}$ and the origin is a saddle point.
- $D\mathbf{f}(a,\sqrt{a}) = \begin{pmatrix} \sqrt{a} & 0 \\ -1 & 2\sqrt{a} \end{pmatrix}$, so $\lambda_1 = \sqrt{a}, \lambda_2 = 2\sqrt{a}$ and the equilibrium is an unstable node.
- $D\mathbf{f}(a,-\sqrt{a}) = \begin{pmatrix} -\sqrt{a} & 0 \\ -1 & -2\sqrt{a} \end{pmatrix}$, so $\lambda_1 = -\sqrt{a}, \lambda_2 = -2\sqrt{a}$ and the equilibrium is an unstable node.

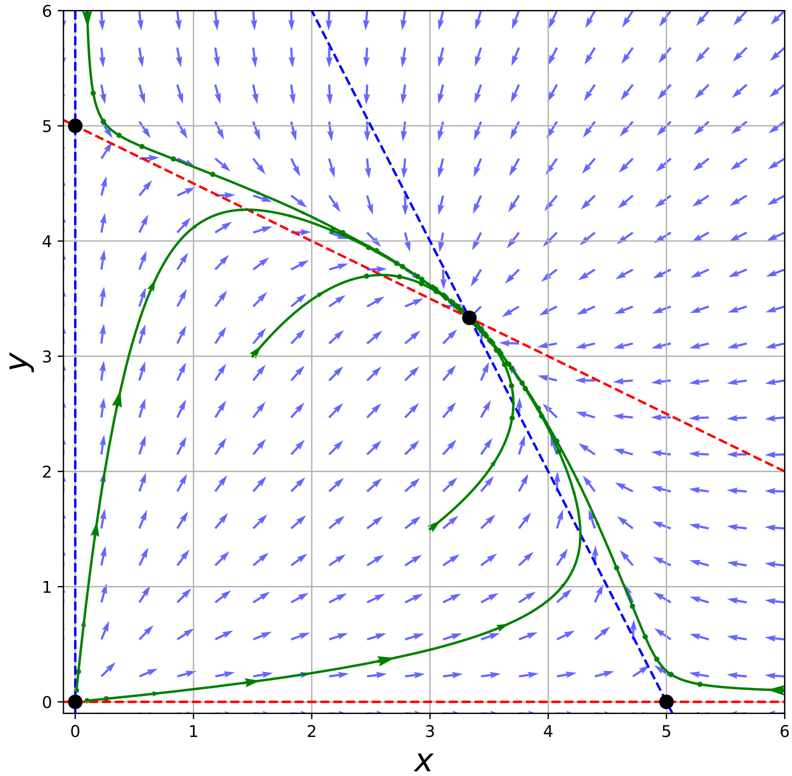


Exercise 12.3 There are four fixed points: $(0,0)$, $(0,5)$, $(5,0)$ and $(10/3, 10/3)$. The Jacobian matrix is

$$D\mathbf{f}(x,y) = \begin{pmatrix} 10-4x-y & -x \\ -y & 10-x-4y \end{pmatrix}.$$

Now, for each fixed point:

- $D\mathbf{f}(0,0) = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$, so $\lambda_1 = 10$ and $\lambda_2 = 10$ and the origin is an unstable node.
- $D\mathbf{f}(0,5) = \begin{pmatrix} 5 & 0 \\ -5 & -10 \end{pmatrix}$, so $\lambda_1 = 5, \lambda_2 = -10$ and the equilibrium is a saddle node.
- $D\mathbf{f}(5,0) = \begin{pmatrix} -10 & -5 \\ 0 & 5 \end{pmatrix}$, so $\lambda_1 = -10, \lambda_2 = 5$ and the equilibrium is a saddle node.
- $D\mathbf{f}(10/3,10/3) = \begin{pmatrix} -20/3 & 10/3 \\ 10/3 & -20/3 \end{pmatrix}$, so $\lambda_1 = -10, \lambda_2 = -10/3$ and the equilibrium is a stable node.



Exercise 12.4 In all cases the Jacobian matrix is

$$D\mathbf{f}(N_1, N_2) = \begin{pmatrix} 1 - 2N_1 - a_{12}N_2 & -a_{12}N_1 \\ -a_{21}N_2 & 1 - 2N_2 - a_{21}N_1 \end{pmatrix}.$$

In all cases we have three fixed points: $(0, 0)$, $(1, 0)$, $(0, 1)$ whose Jacobian matrices are:

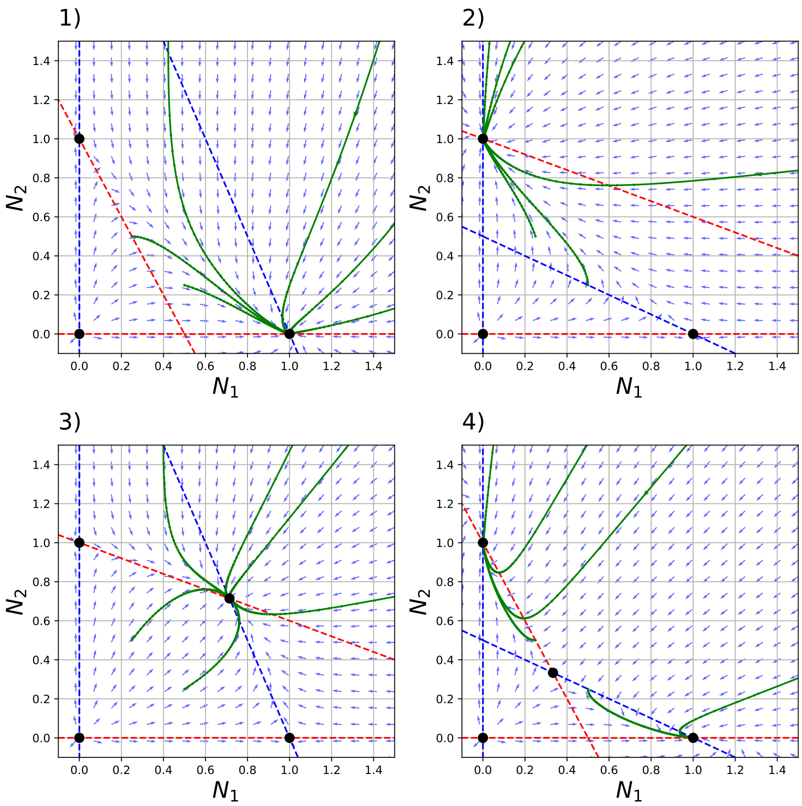
- $D\mathbf{f}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so the origin is always unstable.
- $D\mathbf{f}(1, 0) = \begin{pmatrix} -1 & -a_{12} \\ 0 & 1 - a_{21} \end{pmatrix}$, so this equilibrium will be stable or a saddle point (unstable) depending n the value of a_{21}.
- $D\mathbf{f}(0, 1) = \begin{pmatrix} 1 - a_{12} & 0 \\ -a_{21} & -1 \end{pmatrix}$, so this equilibrium will be stable or a saddle point (unstable) depending n the value of a_{12}.

There is a fourth fixed point $\left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right)$ which is positive (and therefore of interest to us in this population model) if $a_{12} < 1$ and $a_{21} < 1$ (case 3), or if $a_{21} > 1$ and $a_{12} > 1$ (case 4).

1. The fourth fixed point does not matter here. Since $a_{12} < 1$ and $a_{21} > 1$, the point $(1, 0)$ is stable and the point $(0, 1)$ is unstable. So all trajectories go to $(1, 0)$: species 1 outcompetes species 2.
2. The fourth fixed point does not matter here. Since $a_{12} > 1$ and $a_{21} < 1$, the point $(1, 0)$ is unstable and the point $(0, 1)$ is stable. So all trajectories go to $(0, 1)$: species 2 outcompetes species 1.
3. Since both $a_{12} < 1$ and $a_{21} < 1$, both $(1, 0)$ and $(0, 1)$ are saddle points and the fourth fixed point $(10/14, 10/14)$ is stable, since

its Jacobian matrix is $\begin{pmatrix} -10/14 & -4/14 \\ -4/14 & -10/14 \end{pmatrix}$ with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -3/7$. All trajectories go to $(10/14, 10/14)$: there is coexistence of both species.

4. Since both $a_{12} > 1$ and $a_{21} > 1$, both $(1, 0)$ and $(0, 1)$ are saddle points and the fourth fixed point $(1/3, 1/3)$ is a saddle point, since its Jacobian matrix is $\begin{pmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{pmatrix}$ with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1/3$. Trajectories go either to $(1, 0)$ or to $(0, 1)$ depending on the initial conditions.



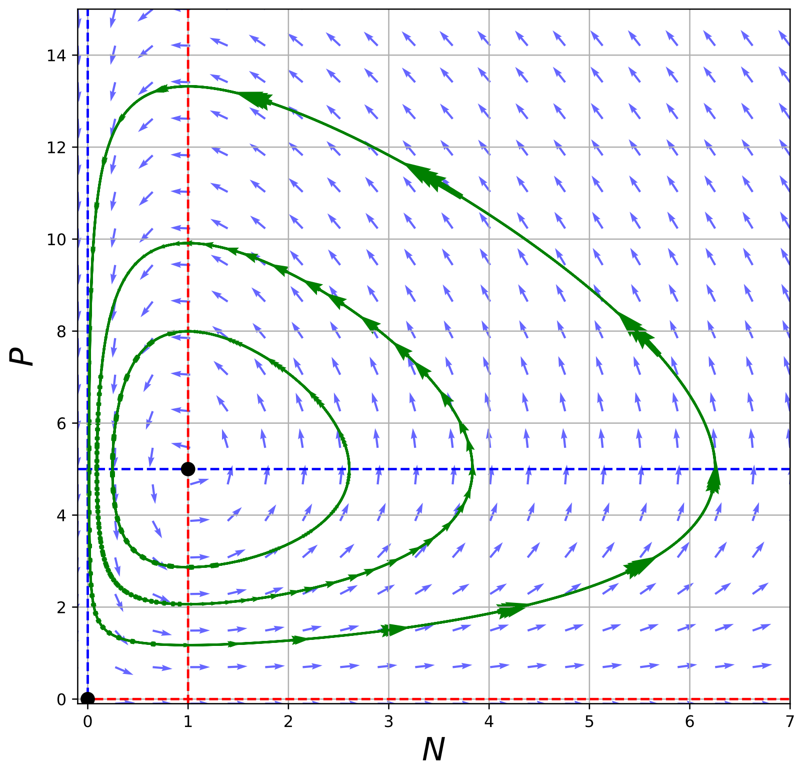
Exercise 12.5 1. The two fixed points are $(0, 0)$ and $(1, 5)$.

2. The Jacobian matrix is

$$D athbf f(N, P) = \begin{pmatrix} 5 - P & -N \\ P & N - 1 \end{pmatrix}.$$

For $(0, 0)$ the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$, so the origin is a saddle point.

3. For $(1, 5)$ the eigenvalues are $\pm\sqrt{5}$, which suggest a center where the trajectories oscillate around the point depending on the initial conditions. This is what happens:



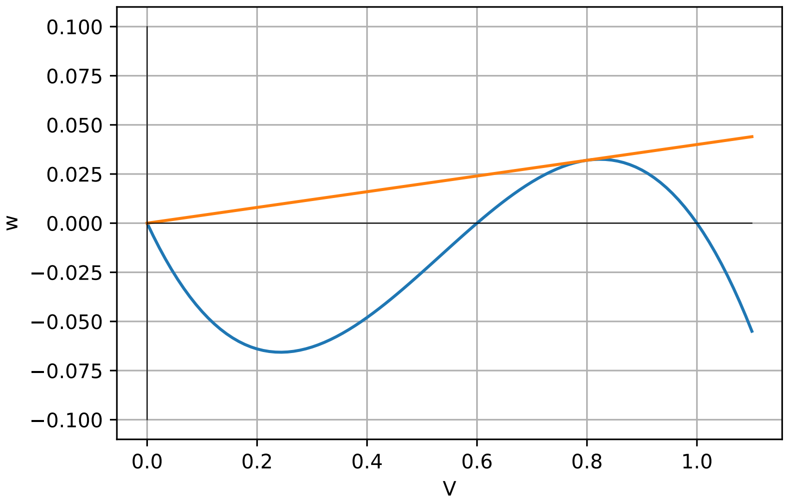
Exercise 12.6 Since we want $-V(V-3/5)(V-1)$ and V/c to intersect, we need to equate the two curves:

$$-V(V-3/5)(V-1) = \frac{V}{c}.$$

Since $V = 0, w = 0$ is always an intersecting point, we can divide by V and we end with the quadratic equation

$$V**2 - \frac{8}{5}V + \frac{3}{5} + \frac{1}{c} = 0 \implies V = \frac{4 \pm \sqrt{\frac{c-25}{c}}}{5}.$$

Since we want this equation to at least have one solution, $c \geq 25$. In particular, the minimal value of c that creates a new intersection (at $x = 4/5$) is $c = 25$.


0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
w
0.0
0.2
0.4
0.6
0.8
1.0
V